NYCMT 2022-2023 HW#2

NYCMT

September 23 - September 30, 2022

1 Solutions

Problem 1. How many subsets S of $\{1, 2, \ldots, 12\}$ satisfy $|S| \geq 2$ and the sum of the two largest elements of S is 15.

Answer: 124
Solution: Once we pick the largest two elements, all the properties of S are satisfied. Thus all that is left to do is determine all possible choices of the remaining elements in S.
We will now do casework on the largest two elements in S.

If $12, 3$ are the largest two elements, there are two possible smaller elements, each of which can be in S or not. Thus there are 2^2 choices of S with $12, 3$ as the two largest elements.
If $11, 4$ are the largest two elements, there are 3 smaller elements, so there are 2^3 sets in this case.
Continuing this process we get 2^4 sets for $10, 5$, 2^5 sets for $9, 6$ and 2^6 sets for $8, 7$.

In total there are $2^2 + 2^3 + 2^4 + 2^5 + 2^6 = 124$ possible choices of S.

Problem 2. Find the interval of real values r such that

$$\lfloor r+\frac{1}{100}\rfloor+\lfloor r+\frac{2}{100}\rfloor+\lfloor r+\frac{3}{100}\rfloor+\cdots+\lfloor r+\frac{70}{100}\rfloor+\lfloor r+\frac{71}{100}\rfloor=619$$

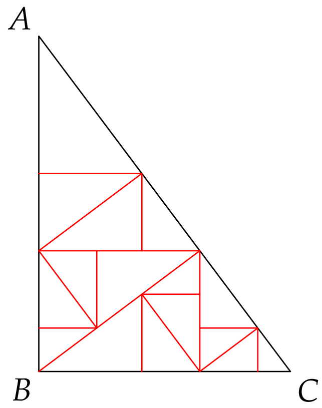
Answer: $r \in [8.79, 8.8)$

Solution: Firstly, note that among all of the terms in our sum, we can have at most 2 distinct values. This is because $[r+\frac{1}{100}, r+\frac{71}{100})$ has length $\frac{7}{10}$, which means that it cannot possibly have a subinterval of length ≥ 1. But in order to achieve at least 3 distinct values, our terms must range from $(k-\epsilon, k+\epsilon)$ (where k is an integer and $\epsilon > 0$ is a real number, which has length > 1.

Now, note that $619 = 71 \cdot 8 + 51$. This implies that since the terms in our sequence are nondecreasing integers, the first $71 - 51 = 20$ terms must be 8, and the next 51 terms must be 9. Then $\lfloor r+\frac{20}{100}\rfloor = 8$ and $\lfloor r+\frac{21}{100}\rfloor = 9$.

The first equation implies that $8 \leq r+\frac{20}{100} < 9 \implies 7.8 \leq r < 8.8$, whereas the second implies that $9 \leq r+\frac{21}{100} < 10 \implies 8.79 \leq r < 9.79$. Combining these bounds gives $8.79 \leq r < 8.8$, or $r \in [8.79, 8.8)$ as desired.

Problem 3. Consider right triangle ABC with $AB = 4$, $BC = 3$, $CA = 5$. Construct the altitude in triangle ABC to the hypotenuse. This splits ABC into 2 right triangles. Construct the altitude to the hypotenuse in these 2 triangles creating 4 triangles. Repeating this process forever a fractal made up of the altitudes is constructed. Does the fractal have finite length? If yes compute it. If not, justify your answer.



Fourth level of the fractal

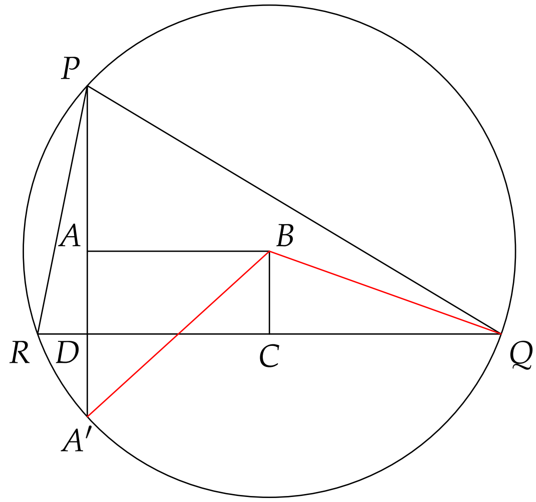
Answer: The fractal has infinite length!
Solution: Let a_n be the sum of the lengths of the segments introduced at the n-th level of the fractal: i.e. the sum of the lengths of the segments that appear at level n of the fractal, but not level $n-1$.

The key claim is that $a_n = \frac{7}{5}a_{n-1}$. This is because when we are adding segments to the fractal, we split each triangle at level $n-1$ into 2 right triangles which satisfy the similarity ratio $3:5$ and $4:5$ with respect to their parent triangle. Then the new altitudes that we add at level n have length $\frac{3}{5}h$ and $\frac{4}{5}h$, where h is the height to the hypotenuse of the parent triangle. Then for each parent triangle from level $n-1$ with height h, 2 new heights are introduced with sum $\frac{7}{5}h$. Summing this over all parent triangles at level $n-1$ gives $a_n = \frac{7}{5}a_{n-1}$.

Now we must check if the sum $\frac{12}{5}(1 + \frac{7}{5} + (\frac{7}{5})^2 + \dots$ converges. Since this is a (nonzero) constant times a geometric series with ratio ≥ 1, this sum diverges, implying the fractal has infinite length as desired.

Problem 4. Rectangle $ABCD$ has $AB = 22$ and $BC = 10$. Assume that there exists some $\triangle PQR$ that has orthocenter A, circumcenter B, C as the midpoint of QR, and D on QR. Find, with proof, the length of QR.

Answer: 56
Solution 1:

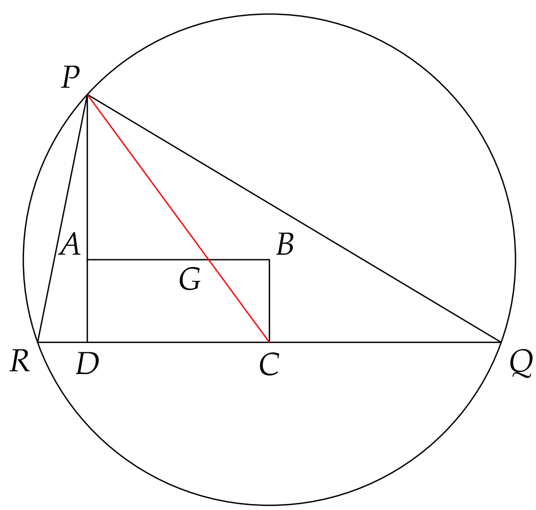


The key claim to this solution is that in any triangle, the reflection of the orthocenter over any side will lie on the circumcircle.
PROOF (click explanation below the diagram to see the proof)
Applying that to the current problem, let's define A' to be the reflection of A over QR. Then the fact above tells us that A' lies on the circumcircle of PQR.

Now, because we know that B is the circumcenter of triangle PQR, $BA' = BR = BQ$. The REALLY nice thing, is that we can now compute BA'!

Note that $BA \perp AA'$ (since $ABCD$ is a rectangle), and $AA' = 2AD = 20$ and $AB = 22$. Using the pythagorean theorem we get $BA' = \sqrt{22^2 + 20^2} = \sqrt{884}$
Now we look at triangle BCQ. $\angle BCQ = 90$ since BC must be the perpendicular bisector of QR. We also know that $BQ = \sqrt{884}$ and $BC = 10$. Thus, we get that $QC = \sqrt{\sqrt{884}^2 - 10^2} = \sqrt{784} = 28$.
Since C is the midpoint of QR, $QR = 2QC = 56$.

Solution 2:



This solution will use the fact that in any triangle, the centroid lies $\frac{2}{3}$ of the way along the segment between orthocenter and circumcenter.
PROOF

Letting the centroid be G, we see that $AG = 2AB$ and $PG = 2GC$. Since $AB \parallel DC$, we know that $\triangle PAG \sim \triangle PDC$, and we know the ratio of similarity is 2 : 3. Thus $PA = 2AD = 20$.

Applying the pythagorean theorem on right triangle $\triangle PAB$, we see that $PB = \sqrt{20^2 + 22^2}$, which is the length of the circumcadius of PQR. The computation from here is the same as in Solution 1.

Problem 5. Given that n has 10 factors the sum of which is 248, compute n.

Answer: 112
Solution: If n can be prime factorized into $p_1^{e_1} \dots p_k^{e_k}$, then n will have $(e_1+1)\dots(e_k+1)$ factors.
Thus, we know that $(e_1+1)\dots(e_k+1) = 10$.

Since $e_i + 1 > 1$, the only possible values of $e_i + 1$ are $e_1 + 1 = 10$ OR $e_1 + 1 = 2$ and $e_2 + 1 = 5$.

Now we want to look at the sum of the factors of n, which can be written as $(1 + p_1 + \cdots + p_1^{e_1})\dots(1 + p_k + \cdots + p_k^{e_k})$.
We know this value is equal to 248, or $8(31)$.

If $e_1 + 1 = 10$ then $e_1 = 9$ and the smallest value of $1 + p_1 + \cdots + p_1^9$ occurs when $p_1 = 2$, and is equal to 1023 which is WAAAAAY too big. Thus we know that $e_1 + 1 = 2$ and $e_2 + 1 = 5$.
Now we know that the number of factors must look like $(1 + p_1)(1 + p_2 + p_2^2 + p_2^3 + p_2^4)$. Note that $1 + p_2 + p_2^2 + p_2^3 + p_2^4$ is always odd and larger than 1. The only odd factor of 248 that is larger than 1 is 31, and so $1 + p_2 + p_2^2 + p_2^3 + p_2^4 = 31$, meaning $p_2 = 2$.
This tells us $1 + p_1 = 8$ so $p_1 = 7$, so $n = 7(2^4) = 112$.