2014 Team Problems

T-1. There exists a digit Y such that, for any digit X, the seven-digit number $\underline{1}\,\underline{2}\,\underline{3}\,\underline{X}\,\underline{5}\,\underline{Y}\,\underline{7}$ is not a multiple of 11. Compute Y.

T-2. A point is selected at random from the interior of a right triangle with legs of length $2\sqrt{3}$ and 4. Let p be the probability that the distance between the point and the nearest vertex is less than 2. Then p can be written in the form $a + \sqrt{b}\pi$, where a and b are rational numbers. Compute (a, b).

T-3. The square $ARML$ is contained in the xy-plane with $A = (0, 0)$ and $M = (1, 1)$. Compute the length of the shortest path from the point $(2/7, 3/7)$ to itself that touches three of the four sides of square $ARML$.

T-4. For each positive integer k, let S_k denote the infinite arithmetic sequence of integers with first term k and common difference k^2. For example, S_3 is the sequence $3, 12, 21, \ldots$. Compute the sum of all k such that 306 is an element of S_k.

T-5. Compute the sum of all values of k for which there exist positive real numbers x and y satisfying the following system of equations.

$$\left\{ \begin{array}{rcl} \log_x y^2 + \log_y x^5 & = & 2k - 1 \\ \log_{x^2} y^5 - \log_{y^2} x^3 & = & k - 3 \end{array} \right.$$

T-6. Let $W = (0, 0)$, $A = (7, 0)$, $S = (7, 1)$, and $H = (0, 1)$. Compute the number of ways to tile rectangle $WASH$ with triangles of area 1/2 and vertices at lattice points on the boundary of $WASH$.

T-7. Compute $\sin^2 4^\circ + \sin^2 8^\circ + \sin^2 12^\circ + \cdots + \sin^2 176^\circ$.

T-8. Compute the area of the region defined by $x^2 + y^2 \leq |x| + |y|$.

T-9. The arithmetic sequences $a_1, a_2, a_3, \ldots, a_{20}$ and $b_1, b_2, b_3, \ldots, b_{20}$ consist of 40 distinct positive integers, and $a_{20} + b_{14} = 1000$. Compute the least possible value for $b_{20} + a_{14}$.

T-10. Compute the ordered triple (x, y, z) representing the farthest lattice point from the origin that satisfies $xy - z^2 = y^2 z - x = 14$.