HMMT November 2017

November 11, 2017

General Round

1. Find the sum of all positive integers whose largest proper divisor is 55. (A proper divisor of n is a divisor that is strictly less than n.)

2. Determine the sum of all distinct real values of x such that
$$|||\cdots||x|+x|\cdots|+x|+x|=1,$$
where there are 2017 x's in the equation.

3. Find the number of integers n with $1 \le n \le 2017$ so that $(n-2)(n-0)(n-1)(n-7)$ is an integer multiple of 1001.

4. Triangle ABC has $AB=10$, $BC=17$, and $CA=21$. Point P lies on the circle with diameter AB. What is the greatest possible area of APC?

5. Given that a, b, c are integers with $abc=60$, and that complex number $\omega \neq 1$ satisfies $\omega^3=1$, find the minimum possible value of $\left|a+b\omega+c\omega^2\right|$.

6. A positive integer n is *magical* if
$$\left\lfloor\sqrt{\lceil\sqrt{n}\rceil}\right\rfloor=\left\lceil\sqrt{\lfloor\sqrt{n}\rfloor}\right\rceil,$$
where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ represent the floor and ceiling function respectively. Find the number of magical integers between 1 and 10,000, inclusive.

7. Reimu has a wooden cube. In each step, she creates a new polyhedron from the previous one by cutting off a pyramid from each vertex of the polyhedron along a plane through the trisection point on each adjacent edge that is closer to the vertex. For example, the polyhedron after the first step has six octagonal faces and eight equilateral triangular faces. How many faces are on the polyhedron after the fifth step?

8. Marisa has a collection of $2^8-1=255$ distinct nonempty subsets of $\{1,2,3,4,5,6,7,8\}$. For each step she takes two subsets chosen uniformly at random from the collection, and replaces them with either their union or their intersection, chosen randomly with equal probability. (The collection is allowed to contain repeated sets.) She repeats this process $2^8-2=254$ times until there is only one set left in the collection. What is the expected size of this set?

9. Find the minimum possible value of
$$\sqrt{58-42x}+\sqrt{149-140\sqrt{1-x^2}}$$
where $-1 \le x \le 1$.

10. Five equally skilled tennis players named Allen, Bob, Catheryn, David, and Evan play in a round robin tournament, such that each pair of people play exactly once, and there are no ties. In each of the ten games, the two players both have a 50% chance of winning, and the results of the games are independent. Compute the probability that there exist four distinct players P_1, P_2, P_3, P_4 such that P_i beats P_{i+1} for $i=1,2,3,4$. (We denote $P_5=P_1$).