2013 Team Problems

T-1. Let x be the smallest positive integer such that $1584 \cdot x$ is a perfect cube, and let y be the smallest positive integer such that xy is a multiple of 1584. Compute y.

T-2. Emma goes to the store to buy apples and peaches. She buys five of each, hands the shopkeeper one \$5 bill, but then has to give the shopkeeper another; she gets back some change. Jonah goes to the same store, buys 2 apples and 12 peaches, and tries to pay with a single \$10 bill. But that's not enough, so Jonah has to give the shopkeeper another \$10 bill, and also gets some change. Finally, Helen goes to the same store to buy 25 peaches. Assuming that the price in cents of each fruit is an integer, compute the least amount of money, in cents, that Helen can expect to pay.

T-3. Circle O has radius 6. Point P lies outside circle O, and the shortest distance from P to circle O is 4. Chord $\overline{AB}$ is parallel to $\overleftrightarrow{OP}$, and the distance between $\overline{AB}$ and $\overleftrightarrow{OP}$ is 2. Compute $PA^2 + PB^2$.

T-4. A *palindrome* is a positive integer, not ending in 0, that reads the same forwards and backwards. For example, $35253, 171, 44$, and 2 are all palindromes, but 17 and 1210 are not. Compute the least positive integer greater than 2013 that *cannot* be written as the sum of two palindromes.

T-5. Positive integers x, y, z satisfy $xy + z = 160$. Compute the smallest possible value of $x + yz$.

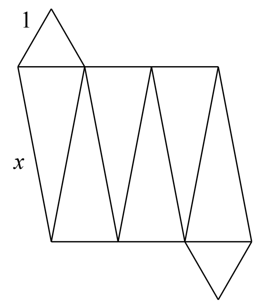
T-6. Compute $\cos^3 \dfrac{2\pi}{7} + \cos^3 \dfrac{4\pi}{7} + \cos^3 \dfrac{8\pi}{7}$.

T-7. In right triangle ABC with right angle C, line ℓ is drawn through C and is parallel to $\overline{AB}$. Points P and Q lie on $\overline{AB}$ with P between A and Q, and points R and S lie on ℓ with C between R and S such that $PQRS$ is a square. Let $\overline{PS}$ intersect $\overline{AC}$ in X, and let $\overline{QR}$ intersect $\overline{BC}$ in Y. The inradius of triangle ABC is 10, and the area of square $PQRS$ is 576. Compute the sum of the inradii of triangles AXP, CXS, CYR, and BYQ.

T-8. Compute the sum of all real numbers x such that

$$\left\lfloor \frac{x}{2} \right\rfloor - \left\lfloor \frac{x}{3} \right\rfloor = \frac{x}{7}.$$

T-9. Two equilateral triangles of side length 1 and six isosceles triangles with legs of length x and base of length 1 are joined as shown below; the net is folded to make a solid. If the volume of the solid is 6, compute x.



T-10. Let $S = \{1, 2, \ldots, 20\}$, and let f be a function from S to S; that is, for all $s \in S, f(s) \in S$. Define the sequence $s_1, s_2, s_3, \ldots$ by setting $s_n = \sum_{k=1}^{20} \underbrace{(f \circ \cdots \circ f)}_{n}(k)$. That is, $s_1 = f(1) + \cdots + f(20)$, $s_2 = f(f(1)) + \cdots + f(f(20))$, $s_3 = f(f(f(1))) + f(f(f(2))) + \cdots + f(f(f(20)))$, etc. Compute the smallest integer p such that the following statement is true: The sequence $s_1, s_2, s_3, \ldots$ must be periodic after a certain point, and its period is at most p. (If the sequence is never periodic, then write ∞ as your answer.)