2011 Team Problems

T-1. If $1, x, y$ is a geometric sequence and $x, y, 3$ is an arithmetic sequence, compute the maximum value of $x + y$.

T-2. Define the sequence of positive integers $\{a_n\}$ as follows:

$$\begin{cases} a_1 = 1; \\ \text{for } n \geq 2,\ a_n \text{ is the smallest possible positive value of } n - {a_k}^2,\ \text{ for } 1 \leq k < n. \end{cases}$$

For example, $a_2 = 2 - 1^2 = 1$, and $a_3 = 3 - 1^2 = 2$. Compute $a_1 + a_2 + \cdots + a_{50}$.

T-3. Compute the base b for which $253_b \cdot 341_b = \underline{7}\,\underline{4}\,\underline{X}\,\underline{Y}\,\underline{Z}_{\,b}$, for some base-$b$ digits X, Y, Z.

T-4. Some portions of the line $y = 4x$ lie below the curve $y = 10\pi \sin^2 x$, and other portions lie above the curve. Compute the sum of the lengths of all the segments of the graph of $y = 4x$ that lie in the first quadrant, below the graph of $y = 10\pi \sin^2 x$.

T-5. In equilateral hexagon $ABCDEF$, $\mathrm{m}\angle A = 2\mathrm{m}\angle C = 2\mathrm{m}\angle E = 5\mathrm{m}\angle D = 10\mathrm{m}\angle B = 10\mathrm{m}\angle F$, and diagonal $BE = 3$. Compute $[ABCDEF]$, that is, the area of $ABCDEF$.

T-6. The *taxicab distance* between points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ is defined as $d(A, B) = |x_A - x_B| + |y_A - y_B|$. Given some $s > 0$ and points $A = (x_A, y_A)$ and $B = (x_B, y_B)$, define the *taxicab ellipse* with foci $A = (x_A, y_A)$ and $B = (x_B, y_B)$ to be the set of points $\{Q \mid d(A, Q) + d(B, Q) = s\}$. Compute the area enclosed by the taxicab ellipse with foci $(0, 5)$ and $(12, 0)$, passing through $(1, -1)$.

T-7. The function f satisfies the relation $f(n) = f(n-1)f(n-2)$ for all integers n, and $f(n) > 0$ for all positive integers n. If $f(1) = \frac{f(2)}{512}$ and $\frac{1}{f(1)} = 2f(2)$, compute $f(f(4))$.

T-8. Frank Narf accidentally read a degree n polynomial with integer coefficients backwards. That is, he read $a_n x^n + \ldots + a_1 x + a_0$ as $a_0 x^n + \ldots + a_{n-1} x + a_n$. Luckily, the reversed polynomial had the same zeros as the original polynomial. All the reversed polynomial's zeros were real, and also integers. If $1 \leq n \leq 7$, compute the number of such polynomials such that $\mathrm{GCD}(a_0, a_1, \ldots, a_n) = 1$.

T-9. Given a regular 16-gon, extend three of its sides to form a triangle none of whose vertices lie on the 16-gon itself. Compute the number of noncongruent triangles that can be formed in this manner.

T-10. Two square tiles of area 9 are placed with one directly on top of the other. The top tile is then rotated about its center by an acute angle θ. If the area of the overlapping region is 8, compute $\sin\theta + \cos\theta$.