Divisibility and Bases Lesson

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1 Divisibility

1.1 Divisibility Rules

What are the divisibility rules for:

- 2?
- 3?
- 4?
- 5?
- 6? How can this be applied to other numbers?
- 8?
- 9?
- 10?
- 11?

1.2 Exercises

1. Is 1748391 divisible by 9?
2. Is 3953851 divisible by 11?

1.3 Proving Divisibility Rules

- Prove the divisibility rule for 2.
- Prove the divisibility rule for 4.
- Prove the divisibility rule for 8.
- Prove the divisibility rule for 3.
- Prove the divisibility rule for 9.
- Prove the divisibility rule for 11.

1.4 Problems

1. Find the largest 3-digit number that is divisible by 22 such that the sum of the units digit and the tens digit is 11.
2. If the five-digit number $ABCDE$ is the 4th power of a whole number and $A+C+E=B+D$, find C.

2 Bases

Base: The number of digits in a number system
Base 10 is what's typically used, which has 10 digits: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

A number $A_n A_{n-1} ... A_2 A_1 A_0$ in base 10 has a value $A_n * 10^n + A_{n-1} * 10^{n-1} + ... + A_2 * 10^2 + A_1 * 10^1 + A_0$.

Likewise, a number $A_n A_{n-1} ... A_2 A_1 A_0$ in base b has a value $A_n * b^n + A_{n-1} * b^{n-1} + ... + A_2 * b^2 + A_1 * b^1 + A_0$.

When a number N is in a certain base b, it is written as N_b.

When writing in a base system which has more than 10 digits, the additional digits will become letters. For example, the digits in base 16 are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F$

2.1 How to convert a number c from base a to base 10

- Consider a number $c = A_n A_{n-1} ... A_2 A_1 A_0$ in base a.
- Write it as $c = A_n * a^n + A_{n-1} * a^{n-1} + ... + A_2 * a^2 + A_1 * a^1 + A_0$.
- Multiply and add (using base 10 rules) until you simplify it down to one number. That number is c in base 10.

2.2 How to convert a number c from base 10 to base b

- Find the largest power of b that is smaller than your number c. Call that power b^n.
- Subtract b^n from c until you get a value less than b^n. Put the number of times you subtracted b^n in the b^n's place.
- Repeat for $b^{n-1}, b^{n-2}, ...b^1, 1$, putting the number of times you subtracted in the correct place.

2.3 How to convert a number c from base a to base b

To do this, you can convert the number from base a to base 10, and then from base 10 to base b, using the two sections above.

2.4 Exercises

1. Convert 83_{10} to base 8.
2. Convert 100_{10} to base 2.
3. Convert 225_7 to base 9.
4. Convert DF_{16} to base 6.

2.5 Problems

1. If N is the base 4 equivalent of 361_{10}, find the square root of N in base 4.
2. Find all solutions to the equation $72_b = 2(27_b)$ or find that there is no solution.
3. Find all solutions to the equation $73_b = 2(37_b)$ or find that there is no solution.
4. ABC and CBA are, respectively, the base 9 and base 7 numerals for the same positive integers. Express this integer in base 10.
5. The three-digit base x numeral $7y3$ is twice the three-digit base x numeral $3y7$. Express $x + y$ as a base 10 numeral.

3 Operations in Different Bases

3.1 Addition and Subtraction

Addition and subtraction are the same, except instead of carrying multiples of 10, you carry multiples of b.

3.2 Problems

1. $101101101_2 + 101000010_2$ (in base 2)
2. $DEF_{16} + ABC_{16}$ (in base 16)
3. $1A4C_{16} - DFE_{16}$ (in base 16)

3.3 Multiplication

When using the multiplication algorithm:

- In the first step, when multiplying digits, find the result in base b, then drop down the one's digit and carry over the "b's digit".
- In the second step, the addition is done as mentioned earlier.

3.4 Problems

1. $9_{19} * 9_{19}$ (in base 19)
2. $10111_2 * 1010_2$ (in base 2)
3. $57_8 * 67_8$ (in base 8)

4 Decimals in Different Bases

4.1 Converting from Base 10 to Base b (non-repeating decimals in base b)

Way 1

- Find the largest power of b that is smaller than your number c. Call that power b^n.
- Subtract b^n from c until you get a value less than b^n. Put the number of times you subtracted b^n in the b^n's place.
- Repeat for $b^{n-1}, b^{n-2}, ...b^1, 1, b^{-1}, b^{-2}, ...$, putting the number of times you subtracted in the correct place. Stop when some subtraction gives you a result of 0.

Way 2

- Multiply c by b repeatedly until you get a whole number. Call the number of times you have multiplied k, and your new number d. Now you know $c = \frac{d}{b^k}$. You can start by converting d to base b.
- Find the largest power of b that is smaller than your number d. Call that power b^n.
- Subtract b^n from d until you get a value less than b^n. Put the number of times you subtracted b^n in the b^n's place.
- Repeat for $b^{n-1}, b^{n-2}, ...b^1, 1$, putting the number of times you subtracted in the correct place.
- Now, divide d by b^k to get your result c by shifting the decimal place to the left k times.

Note: These methods will only work if c is a non-repeating decimal in base b.

An extreme example: $\frac{1}{49} = 0.\overline{020408163265306122448979591836734693877551}_{10} = 0.01_7$.

4.2 Problems

1. Express 9.25_{10} in base 2.
2. Express 3.65625_{10} in base 2.
3. Express $11.\bar{5}_{10}$ in base 3.

4.3 Converting from base b to base 10 (non-repeating decimals in base b)

- Consider a number $c = A_n A_{n-1}...A_2 A_1 A_0.A_{-1}A_{-2}...A_{-m}$ in base a.
- Write it as $c = A_n * a^n + A_{n-1} * a^{n-1} + ... + A_2 * a^2 + A_1 * a^1 + A_0 + a^{-1} * A_{-1} + a^{-2} * A_{-2} + ... + a^{-m} * A_{-m}$.
- Multiply and add (using base 10 rules) until you simplify it down to one number. That number is c in base 10.

4.4 Converting from base b to base 10 (repeating decimals in base b)

4.4.1 Exercises

1. Write $0.\overline{1}_{10}$ as a fraction.
2. Write $0.\overline{01}_{10}$ as a fraction.
3. Write $0.0\overline{1}_{10}$ as a fraction.

4.4.2 Process

Repeating fractions work similarly in different bases.

Steps:

- Convert the integral part using techniques discussed earlier, and subtract it out.
- Convert the non-repeating fractional part using techniques discussed earlier, and subtract it out.
- You will now have something in the form of $0.00...00\overline{A_1 A_2 ... A_n}_b$.
- Let $k = A_1 A_2 ... A_n$, and let m be the number of zeroes preceding the repeating fraction.
- Then, the repeating fraction is equal to $\frac{k}{b^m * (b^n - 1)}$
- The answer will be the sum of the integral part, the non-repeating fractional part, and the repeating fraction.

4.5 Problems

1. Write $64.\overline{12}_7$ as a base 10 decimal.
2. Write $11.\overline{36}_{11}$ as a base 10 decimal.

4.6 Converting from base 10 to base b (repeating decimals in base b)

- Convert the integral part using the techniques discussed earlier, and subtract it out.
- Write the fractional part as a fraction.
- Multiply both sides until the denominator is equal to $b^m * (b^n - 1)$, where m and n are non-negative integers. Call the fraction $\frac{k}{b^m*(b^n-1)}$.
- Factor out $1/b^m$. For now, consider $\frac{k}{b^n-1}$.
- Write as a decimal. It will be equal to (Integer factored out).(k mod $b^n - 1$ repeating, with n digits repeated)
- Divide by b^m.
- Add the integral part.

4.7 Problems

1. Write 10.2_{10} in base 6.
2. Write 11.23_{10} in base 5.

5 AMC/AIME Problems

1. The base-ten representation for 19! is $121, 6T5, 100, 40M, 832, H00$, where T, M, and H denote digits that are not given. What is $T + M + H$? (2019 AMC 10B)

   (A) 3 **(B)** 8 **(C)** 12 **(D)** 14 **(E)** 17

2. The number n can be written in base 14 as $\underline{a}\underline{b}\underline{c}$, can be written in base 15 as $\underline{a}\underline{c}\underline{b}$, and can be written in base 6 as $\underline{a}\underline{c}\underline{a}\underline{c}$, where $a > 0$. Find the base-10 representation of n. (2018 AIME I)
3. A positive integer N has base-eleven representation $\underline{a}\,\underline{b}\,\underline{c}$ and base-eight representation $\underline{1}\,\underline{b}\,\underline{c}\,\underline{a}$, where a, b, and c represent (not necessarily distinct) digits. Find the least such N expressed in base ten. (2020 AIME I)

6 Divisibility Rules in Other Bases

There are many divisibility rules in base 10. However, divisibility rules are not unique to this base.

6.1 Divisibility rule for $n - 1$ in base n

The divisibility rule for 9 in base 10 can be extended to all bases. Here is the proof:

- Consider a number which can be written in base n as $x = A_j A_{j-1} A_{j-2} ... A_2 A_1 A_{0n}$.
- This number's value in base 10 is $A_j * n^j + A_{j-1} * n^{j-1} + A_{j-2} * n^{j-2} + ... + A_2 n^2 + A_1 n + A_0$.
- For any natural number k, $n - 1 | n^k - 1$.
- $A_j * n^j + A_{j-1} * n^{j-1} + A_{j-2} * n^{j-2} + ... + A_2 n^2 + A_1 n + A_0 = (A_j * (n^j - 1) + A_{j-1} * (n^{j-1} - 1) + A_{j-2} * (n^{j-2} - 1) + ... + A_2(n^2 - 1) + A_1(n - 1)) + (A_j + A_{j-1} + A_{j-2} + ... + A_2 + A_1 + A_0)$
- $A_j * n^j + A_{j-1} * n^{j-1} + A_{j-2} * n^{j-2} + ... + A_2 n^2 + A_1 n + A_0 = (n-1)Y + (A_j + A_{j-1} + A_{j-2} + ... + A_2 + A_1 + A_0)$
- Therefore, x is divisible by $n - 1$ if the sum of its digits in base n is divisible by $n - 1$.

This rule will work for all divisors of $n - 1$, because the term divisible by $n - 1$ will also have no effect for the divisors.

6.2 Divisibility rule for $n + 1$ in base n

The divisibility rule for 11 in base 10 can be extended to all bases. Here is the proof:

- Consider a number which can be written in base n as $x = A_j A_{j-1} A_{j-2} ... A_2 A_1 A_{0n}$. I'll assume j is even, but A_j can be zero if A_{j-1} is nonzero.
- This number's value in base 10 is $A_j * n^j + A_{j-1} * n^{j-1} + A_{j-2} * n^{j-2} + ... + A_2 n^2 + A_1 n + A_0$.
- For any natural number k, $n + 1 | n^k + 1$ if k is odd and $n + 1 | n^k - 1$ if k is even.
- $A_j * n^j + A_{j-1} * n^{j-1} + A_{j-2} * n^{j-2} + ... + A_2 n^2 + A_1 n + A_0 = (A_j * (n^j - 1) + A_{j-1} * (n^{j-1} + 1) + A_{j-2} * (n^{j-2} - 1) + ... + A_2(n^2 - 1) + A_1(n + 1)) + (A_j - A_{j-1} + A_{j-2} + ... + A_2 - A_1 + A_0)$
- $A_j * n^j + A_{j-1} * n^{j-1} + A_{j-2} * n^{j-2} + ... + A_2 n^2 + A_1 n + A_0 = (n+1)Y + (A_j - A_{j-1} + A_{j-2} + ... + A_2 - A_1 + A_0)$
- Therefore, x is divisible by $n + 1$ if the sum of its digits in base n is divisible by $n + 1$.

This rule will work for all divisors of $n + 1$, because the term divisible by $n + 1$ will also have no effect for the divisors.

6.3 Exercises

1. Is 111010111_2 divisible by 3?
2. Is 532014_6 divisible by 5, 7, both, or none?
3. Is $4920AB_{12}$ divisible by 11, 13, both, or none?

6.4 Grouping Digits in Base 10 to find more divisibility rules

These rules will also work on base 100 and base 1000. This means there is a divisibility rule for 101 in base 100, and a divisibility rule for 999 (and therefore 37, 27) and 1001 (and therefore 13) in base 1000.

A number can be easily converted from base 10 to base 100 or base 1000 by grouping digits into twos or threes, respectively, starting from the right.

Examples:
$190898 \rightarrow 19 - 08 - 98_{100}$
$190898 \rightarrow 190 - 898_{1000}$

This opens up some new divisibility rules:

- 101: The difference between alternating sums of pairs of digits (starting from the right) is a multiple of 101.
- 13: The difference between alternating sums of triplets of digits (starting from the right) is a multiple of 13.
- 27: The sum of triplets of digits (starting from the right) is a multiple of 27.
- 37: The sum of triplets of digits (starting from the right) is a multiple of 37.

These rules get to the point where dividing might be easier. However, they can be useful in certain circumstances.

6.5 Exercises

Determine if the following numbers are divisible by 13, 27, 37, 101:

1. 18930834
2. 12001986

7 Problems

1. Hexadecimal (base-16) numbers are written using numeric digits 0 through 9 as well as the letters A through F to represent 10 through 15. Among the first 1000 positive integers, there are n whose hexadecimal representation contains only numeric digits. What is the sum of the digits of n? (2015 AMC 10A)

   (A) 17 **(B)** 18 **(C)** 19 **(D)** 20 **(E)** 21

2. Find the sum of all positive integers $b < 1000$ such that the base-b integer 36_b is a perfect square and the base-b integer 27_b is a perfect cube. (2018 AIME II)

3. The first 2007 positive integers are each written in base 3. How many of these base-3 representations are palindromes? (A palindrome is a number that reads the same forward and backward.) (2007 AMC 12B)

   (A)100 **(B)**101 **(C)**102 **(D)**103 **(E)**104

4. Find the number of positive integers less than or equal to 2017 whose base-three representation contains no digit equal to 0. (2017 AIME II)

5. A rational number written in base eight is $\underline{ab}.\underline{cd}$, where all digits are nonzero. The same number in base twelve is $\underline{bb}.\underline{ba}$. Find the base-ten number $\underline{abc}$. (2017 AIME I)

6. Call a positive integer N a 7-10 double if the digits of the base-7 representation of N form a base-10 number that is twice N. For example, 51 is a 7-10 double because its base-7 representation is 102. What is the largest 7-10 double? (2001 AIME I)