NYC Math Team: October Test

October 2020

December 7, 2020

Problems

Problem 1. How many permutations of N, Y, C, M, T are there such that there is exactly one letter between N and Y?

Problem 2. If x and y are real numbers such that $x^3 = \frac{16}{y^2}$ and $y^3 = \frac{8}{x^2}$, then compute $(x+y)^5$.

Problem 3. Find the largest positive integer n such that n^3 leaves a remainder of 1 when divided by $n-3$.

Problem 4. In $\triangle ABC$, let $AB = 13, BC = 14$, and $CA = 15$. Let D be the foot of the altitude from A to BC. Let E and F be points on segments AB and AC such that $AEDF$ is a parallelogram. Compute the area of this parallelogram.

Problem 5. How many solutions for x in the range $(0, 2020)$ are there such that

$$\lfloor x \rfloor^2 + 2020\{x\}^2 = x^2?$$

(Here, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x, while $\{x\}$ denotes $x - \lfloor x \rfloor$.)

Problem 6. Let $f(x) = \gcd(x^2, 2x+6)$ for all positive integers x and let k be the maximum possible value of $f(x)$. Suppose n is the fifth smallest positive integer such that $f(n) = k$. Find the ordered pair (k, n).

Problem 7. Let $\triangle O_1XY$ be isosceles, with $O_1X = O_1Y$ and $\angle XO_1Y \neq 90^\circ$. Let O_2 be the circumcenter of $\triangle O_1XY$, and let O_3 be the circumcenter of $\triangle O_2XY$. If $\angle XO_1Y = \angle XO_3Y = \theta$, find the sum of all possible values of θ in degrees.

Problem 8. How many integers between 1 and 1023, inclusive, are there such that when expressed in binary, every 0 is adjacent to another 0, and every 1 is adjacent to another 1? (For example, the binary integer 1100_2 has this property, but the binary integer 1010_2 does not.)

Problem 9. Call a positive integer m *ridonkulous* if for all positive integers n, the set

$$\{5040n, 5040n+1, 5040n+2, \ldots, 5040n+5040\}$$

contains a multiple of m. How many ridonkulous numbers are there?

Problem 10. Let a, b, and c be the roots of $x^3 - 9x - 2020 = 0$. Compute

$$(a^2 - bc)(b^2 - ca)(c^2 - ab).$$

Problem 11. In $\triangle ABC$, the midpoints of AB and AC are M and N respectively. Let BN and CM intersect at G. If $AMGN$ is cyclic, $BC = 6$, and $\angle BAC = 30^\circ$, compute $(AB + AC)^2$.

Problem 12. Compute

$$\sum_{a=0}^{4}\sum_{b=0}^{4}\sum_{c=0}^{4} \frac{(a+b+c)!}{a!b!c!} \cdot \frac{(12-a-b-c)!}{(4-a)!(4-b)!(4-c)!}.$$

Problem 13. In $\triangle ABC$, $AB = 5, BC = 7$, and $CA = 8$. Let the perpendicular bisector of BC intersect lines AB and AC at X and Y. Let the circumcircle of $\triangle ABY$ be ω_1 and the circumcircle of $\triangle ACX$ be ω_2. Let ω_1 and ω_2 intersect at a point $T \neq A$. Let the tangents to ω_1 and ω_2 at T intersect BC at U and V, respectively. Compute UV.

Problem 14. Compute $\displaystyle\prod_{k=1}^{8} \cos\left(\frac{k^2\pi}{17}\right)$.

Problem 15. Let S be the set of positive integer divisors of 1000. How many functions $f : S \longrightarrow S$ are there such that for any $x, y \in S$,

$$\gcd((f(x), f(y)) = f(\gcd(x, y))?$$

Problem 16. How many ways are there to choose integers x, y, z between 1 and 103, inclusive, such that $x^2 + y^2 + z^2 - xyz \equiv 4 \pmod{103}$?

Solutions

Problem 1

How many permutations of N, Y, C, M, T are there such that there is exactly one letter between N and Y?

Solution. This means that the N and the Y must be separated by exactly one letter. There are exactly 6 ways to do this: three choices for which spots they fill up and two choices for the order in which they go. The rest of the letters can go in any order in the three remaining spots. Thus, the answer is $6 \cdot 3! = \boxed{36}$.

Problem 2

If x and y are real numbers such that $x^3 = \frac{16}{y^2}$ and $y^3 = \frac{8}{x^2}$, then compute $(x+y)^5$.

Solution. Rearrange the equations into

$$x^3y^2 = 16 \text{ and } x^2y^3 = 8.$$

Multiplying these together give $x^5y^5 = 2^7$, so $xy = 2^{7/5}$. Now adding our two equations together give

$$\begin{aligned} x^2y^2(x+y) &= 24 \\ 2^{14/5}(x+y) &= 2^3 \cdot 3 \\ x+y &= 2^{1/5} \cdot 3 \\ (x+y)^5 &= 2 \cdot 3^5, \end{aligned}$$

so $(x+y)^5 = \boxed{486}$.

Problem 3

Find the largest positive integer n such that n^3 leaves a remainder of 1 when divided by $n - 3$.

Solution. Note that $n-3 \mid n^3-27$ since the right side is a difference of cubes. Since we also need $n-3 \mid n^3-1$ by the problem statement, we must have

$$n - 3 \mid (n^3 - 1) - (n^3 - 27) = 26.$$

The largest such n is $\boxed{29}$.

Problem 4

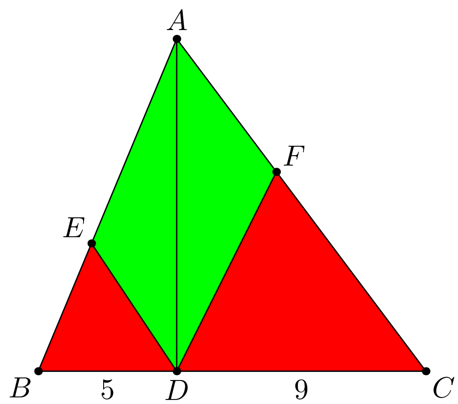
In $\triangle ABC$, let $AB = 13, BC = 14$, and $CA = 15$. Let D be the foot of the altitude from A to BC. Let E and F be points on segments AB and AC such that $AEDF$ is a parallelogram. Compute the area of this parallelogram.

Solution. By Heron's formula, the area of $\triangle ABC$ is

$$\sqrt{21 \cdot 6 \cdot 7 \cdot 8} = 84.$$

Then the height from A to BC multiplied by BC must be equal to $2 \cdot 84 = 168$, so the height from A to BC is 12. Then $BD = 5$ and $CD = 9$ from well-known right triangles.

(The above facts are all well-known about a $13 - 14 - 15$ triangle.)



Note that the area of $AEDF$ is the area of ABC minus the two areas of the red triangles. But the left red triangle is $\left(\frac{5}{14}\right)^2$ the area of the original triangle, and the right red triangle is $\left(\frac{9}{14}\right)^2$ the area of the original triangle. Thus, the area of the green region is

$$\left(1 - \frac{25}{196} - \frac{81}{196}\right)[ABC] = \frac{90}{196} \cdot 84 = \frac{45}{98} \cdot 84 = \boxed{\frac{270}{7}}.$$

Problem 5

How many solutions for x in the range $(0, 2020)$ are there such that

$$\lfloor x \rfloor^2 + 2020\{x\}^2 = x^2?$$

(Here, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x, while $\{x\}$ denotes $x - \lfloor x \rfloor$.)

Solution. Replace x with $\lfloor x \rfloor + \{x\}$ on the right side and expand. We are left with

$$\begin{aligned} \lfloor x \rfloor^2 + 2020\{x\}^2 &= x^2 \\ \lfloor x \rfloor^2 + 2020\{x\}^2 &= \lfloor x \rfloor^2 + 2\lfloor x \rfloor \{x\} + \{x\}^2 \\ 2019\{x\}^2 &= 2\lfloor x \rfloor \{x\} \\ \{x\}(2019\{x\} - 2\lfloor x \rfloor) &= 0 \end{aligned}$$

Thus, either $\{x\} = 0$ or $\frac{2}{2019}\lfloor x \rfloor = \{x\}$. The former produces solutions of all integers in the range $(0, 2020)$, of which there are 2019.

We now look at the latter equation. Note that $\{x\} < 1$, so $\lfloor x \rfloor < \frac{2019}{2}$. After determining $\lfloor x \rfloor$, the value of x is uniquely determined as we can determine $\{x\}$. There are 1009 possible values of $\lfloor x \rfloor$, so there are 1009 possible numbers in this case.

The answer is then $2019 + 1009 = \boxed{3028}$.

Problem 6

Let $f(x) = \gcd(x^2, 2x+6)$ for all positive integers x and let k be the maximum possible value of $f(x)$. Suppose n is the fifth smallest positive integer such that $f(n) = k$. Find the ordered pair (k, n).

Solution. Note that

$$f(x) = \gcd(x^2, 2x+6) \le 2 \cdot \gcd(x^2, x+3) \le 2 \cdot \gcd(x^2 - (x+3)(x-3), x+3) = 2 \cdot \gcd(9, x+3) \le 18.$$

Equality holds when $9 \mid x+3$ and $2 \mid x$, which if and only if $x \equiv 6 \pmod{18}$ by CRT. The fifth smallest integer satisfying this is $5 \cdot 18 - 12 = 78$. Thus, $k = 18$ and $n = 78$, giving $\boxed{(18, 78)}$.

Problem 7

Let $\triangle O_1XY$ be isosceles, with $O_1X = O_1Y$ and $\angle XO_1Y \neq 90°$. Let O_2 be the circumcenter of $\triangle O_1XY$, and let O_3 be the circumcenter of $\triangle O_2XY$. If $\angle XO_1Y = \angle XO_3Y = \theta$, find the sum of all possible values of θ in degrees.

Solution. Suppose $\angle XO_iY = \alpha$, for some $i \in \{1, 2\}$.
If $\alpha < 90°$, then $\angle XO_{i+1}Y = 2\alpha$.
If $\alpha > 90°$, then $\angle XO_{i+1}Y = 360° - 2\alpha$.
This leaves us with one of four possibilities:

$$\begin{aligned}
&\theta \longrightarrow 2\theta \longrightarrow 4\theta = \theta \\
&\theta \longrightarrow 2\theta \longrightarrow 360° - 4\theta = \theta \\
&\theta \longrightarrow 360° - 2\theta \longrightarrow 720° - 4\theta = \theta \\
&\theta \longrightarrow 360° - 2\theta \longrightarrow 4\theta - 360° = \theta
\end{aligned}$$

The first case yields $\theta = 0°$, which is not valid.
The second case yields $\theta = 72°$. We can check that our moves are indeed valid: $72° \longrightarrow 144° \longrightarrow 72°$ are both valid moves.
The third case yields $\theta = 144°$. We can check that our moves are indeed valid: $144° \longrightarrow 72° \longrightarrow 144°$ are both valid moves.
The last case yields $\theta = 120°$. We can check that our moves are indeed valid: $120° \longrightarrow 120° \longrightarrow 120°$ are both valid moves.
The answer is $72° + 144° + 120° = \boxed{336°}$.

Problem 8

How many integers between 1 and 1023, inclusive, are there such that when expressed in binary, every 0 is adjacent to another 0, and every 1 is adjacent to another 1? (For example, the binary integer 1100_2 has this property, but the binary integer 1010_2 does not.)

Solution. Let a_n denote the number of such numbers which have exactly n digits when written in binary. We seek $a_1 + a_2 + a_3 + \cdots + a_{10}$. We now compute a recursive formula for a_n.
Note that the first digit must be a 1, so the next digit must also be a 1. Let the first $k \geq 2$ digits be 1. Then the remaining $n - k$ digits have the same rules as before, except they are required to start with a 0. If we interchange the roles of 1s and 0s, we see that there are a_{n-k} ways to finish. Thus, we have

$$a_n = a_{n-2} + a_{n-3} + \cdots + a_2 + a_1 + a_0,$$

where we define $a_0 = 1$. Similarly, we have

$$a_{n+1} = a_{n-1} + a_{n-2} + \cdots + a_2 + a_1 + a_0.$$

Subtracting the two, we obtain

$$a_{n+1} = a_n + a_{n-1}.$$

Since $a_0 = 1$ and $a_1 = 0$, we see that $a_n = F_{n-1}$, where F_k is the kth Fibonacci number.
Now

$$a_1 + a_2 + \cdots + a_{10} = a_{12} - a_0 = F_{11} - 1 = 89 - 1 = \boxed{88}.$$

Problem 9

Call a positive integer m *ridonkulous* if for all positive integers n, the set

$$\{5040n, 5040n+1, 5040n+2, \ldots, 5040n+5040\}$$

contains a multiple of m. How many ridonkulous numbers are there?

It is clear that all positive integers at most 5040 are ridonkulous. We claim that the ridonkulous numbers that are greater than 5040 are precisely $5040+d$, where $d \mid 5040$.

To prove this claim, first we show that these are all ridonkulous. Suppose $d \mid 5040$. Let $d' = \frac{5040}{d}$. Since $d \mid 5040+d$, we just need $5040+d$ to divide an element in the set

$$\{5040n, 5040n+d, 5040n+2d, \ldots, 5040n+5040\}.$$

We can divide everything by d, so we need $d'+1$ to divide a number in the set

$$\{d'n, d'n+1, d'n+2, \ldots, d'n+d'\}.$$

This is a set of $d'+1$ consecutive positive integers, so $d'+1$ must divide one of them.

We must now prove the converse. Let our number be $5040+k$ with $k \nmid 5040$. Let $d = \gcd(k, 5040) < k$. Let $d' = \frac{5040}{d}$ and $k' = \frac{k}{d} > 1$. Since $d \mid 5040+k$, we must have $5040+k$ dividing a number in the set

$$\{5040n, 5040n+d, 5040n+2d, \ldots, 5040n+5040\}$$

for all n. We can divide everything by d, so we need $d'+k'$ to divide a number in the set $\{d'n, d'n+1, d'n+2, \cdots, d'n+d'\}$ for all n. Note that $\gcd(d'+k', d') = \gcd(d', k') = 1$ since otherwise we could have increased d. Thus, we can choose an n such that $d'n \equiv 1 \pmod{d'+k'}$. For this n, since $k' > 1$, no number in the set is a multiple of $d'+k'$, so $5040+k$ isn't ridonkulous.

Thus, the claim is proven. There are 5040 integers at most 5040, and there are 60 divisors of $5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$, so the answer is

$$5040 + 60 = \boxed{5100}.$$

Problem 10

Let a, b, and c be the roots of $x^3 - 9x - 2020 = 0$. Compute

$$(a^2 - bc)(b^2 - ca)(c^2 - ab).$$

Solution. By Vieta's, we have $abc = 2020$. Then our expression reduces to

$$\left(a^2 - \frac{2020}{a}\right)\left(b^2 - \frac{2020}{b}\right)\left(c^2 - \frac{2020}{c}\right) = \frac{(a^3 - 2020)(b^3 - 2020)(c^3 - 2020)}{abc}.$$

However, $a^3 - 2020 = 9a$, $b^3 - 2020 = 9b$, and $c^3 - 2020 = 9c$. Thus, our expression reduces to

$$\frac{729abc}{abc} = \boxed{729}.$$

Remark. $(a^2 - bc)(b^2 - ca)(c^2 - ab) + (ab + bc + ca)^3 = abc(a + b + c)^3$

Problem 11

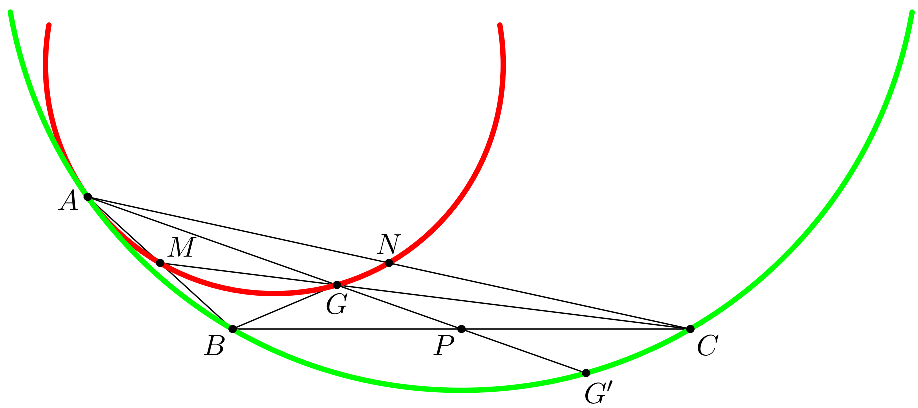
In $\triangle ABC$, the midpoints of AB and AC are M and N respectively. Let BN and CM intersect at G. If $AMGN$ is cyclic, $BC = 6$, and $\angle BAC = 30^\circ$, compute $(AB + AC)^2$.

Solution (Reflection Across Midpoint)**.** Note that G is just the centroid of $\triangle ABC$. Let the midpoint of BC be P. We need $\angle BGC = 150^\circ$, so the reflection of G over P must lie on (ABC). Then by power of a point, $\frac{m_a^2}{3} = \frac{a^2}{4} = 9$, where m_a is the length of the A-median. Thus, $m_a = 3\sqrt{3}$.

By Law of Cosines on $\angle BAC$ of $\triangle ABC$, we have $b^2 + c^2 - \sqrt{3}bc = 36$. By the median formula, we have

$$\frac{2(b^2 + c^2) - 36}{4} = m_a^2 = 27,$$

or $b^2 + c^2 = 72$. Then $bc = 12\sqrt{3}$, so $(b + c)^2 = 72 + 24\sqrt{3}$.



Solution (Power of a Point)**.** Notice that the centroid divides a median in a $2:1$ ratio. This means $BG : GN = 2 : 1$ and $CG : GM = 2 : 1$. By power of a point, $BG \cdot BN = BM \cdot BA$, which can be rewritten as $BG \cdot 3BG = BM \cdot 2BM$. Similarly, $CG \cdot 3CG = CM \cdot 2CM$. Thus, if $GN = x$, then

$$\begin{aligned} BG &= 2x, \\ BM &= x\sqrt{3}, \\ AM &= x\sqrt{3}. \end{aligned}$$

Similarly, if $GM = y$, then

$$\begin{aligned} CG &= 2y, \\ CN &= y\sqrt{3}, \\ AN &= y\sqrt{3}. \end{aligned}$$

From Law of Cosines on $\triangle ABC$, we see that

$$6^2 = 12x^2 + 12y^2 - 12xy\sqrt{3}, \text{ or } 3 = x^2 + y^2 - xy\sqrt{3}.$$

Since $\angle BGC = 150^\circ$ we see that

$$6^2 = 4x^2 + 4y^2 + 4xy\sqrt{3}, \text{ or } 9 = x^2 + y^2 + xy\sqrt{3}.$$

Thus, $x^2 + y^2 = 6$ and $xy = \sqrt{3}$. Now $(AB + AC)^2 = 12(x + y)^2$, which is simply $12(x^2 + 2xy + y^2)$, or $\boxed{72 + 24\sqrt{3}}$.

Problem 12

Compute

$$\sum_{a=0}^{4}\sum_{b=0}^{4}\sum_{c=0}^{4}\frac{(a+b+c)!}{a!b!c!}\cdot\frac{(12-a-b-c)!}{(4-a)!(4-b)!(4-c)!}.$$

Solution (Combinatorial Interpretation). For a fixed (a,b,c), note that

$$\frac{(a+b+c)!}{a!b!c!}$$

represents the number of paths from $(0,0,0)$ to (a,b,c), while

$$\frac{(12-a-b-c)!}{(4-a)!(4-b)!(4-c)!}$$

represents the number of paths from (a,b,c) to $(4,4,4)$. Thus, the product

$$\sum_{a=0}^{4}\sum_{b=0}^{4}\sum_{c=0}^{4}\frac{(a+b+c)!}{a!b!c!}\cdot\frac{(12-a-b-c)!}{(4-a)!(4-b)!(4-c)!}$$

is the number of paths from $(0,0,0)$ to $(4,4,4)$ that pass through (a,b,c). Then the summation is

$$\sum_{a=0}^{4}\sum_{b=0}^{4}\sum_{c=0}^{4}[\text{Number of paths through }(a,b,c)]=\sum_{\text{path}}[\text{Number of lattice points on path}]=\binom{12}{4,4,4}\cdot 13.$$

This is equal to

$$\frac{13!}{24^3}=\frac{13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{24\cdot 24\cdot 24}=\frac{13\cdot \cancel{12}\cdot 11\cdot 10\cdot 9\cdot \cancel{8}\cdot 7\cdot \cancel{6}\cdot 5\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1}{\cancel{24}\cdot \cancel{24}\cdot \cancel{24}}$$
$$=(13\cdot 11\cdot 7)(10\cdot 9\cdot 5)=1001\cdot 450=\boxed{450450}.$$

Solution (Algebra). Observe that the numerator can alternatively be expressed as

$$(12-a-b-c)!(a+b+c)!=\frac{12!}{\binom{12}{a+b+c}}$$

and the denominator can be expressed as

$$a!b!c!(4-a)!(4-b)!(4-c)!=\frac{4!}{\binom{4}{a}}\cdot\frac{4!}{\binom{4}{b}}\cdot\frac{4!}{\binom{4}{b}}.$$

Thus, we can rewrite the fraction in the sum as

$$\frac{(a+b+c)!}{a!b!c!}\cdot\frac{(12-a-b-c)!}{(4-a)!(4-b)!(4-c)!}=\frac{12!\binom{4}{a}\binom{4}{b}\binom{4}{c}}{(4!)^3\binom{12}{a+b+c}}.$$

We compute the sum by fixing the value of $a+b+c$. The value of $a+b+c$ ranges from 0 to 12 (since a,b, and c each range from 0 to 4). The sum becomes

$$\sum_{s=0}^{12}\sum_{a+b+c=s}\frac{12!\binom{4}{a}\binom{4}{b}\binom{4}{c}}{(4!)^3\binom{12}{a+b+c}}=\sum_{s=0}^{12}\sum_{a+b+c=s}\frac{12!\binom{4}{a}\binom{4}{b}\binom{4}{c}}{(4!)^3\binom{12}{s}}.$$

Now, we attempt to compute the inner sum using a combinatorial argument.

Consider a committee of 12 people, and suppose we choose s people from this committee. This can be done in $\binom{12}{s}$ ways. On the other hand, we can split up the committee into three groups of 4. We choose a people from the first group, b people from the second group, and c people from the third group such that $0 \leq a, b, c \leq 4$ and $a + b + c = 12$. This can be done in $\binom{4}{a}\binom{4}{b}\binom{4}{c}$ ways for each choice of (a, b, c) satisfying these conditions. This means

$$\sum_{a+b+c=s} \binom{4}{a}\binom{4}{b}\binom{4}{c} = \binom{12}{s}.$$

(Alternatively, one can look at the coefficient of x^s when $(1 + x)^1 2 = [(1 + x)^4]^3$ is expanded.)

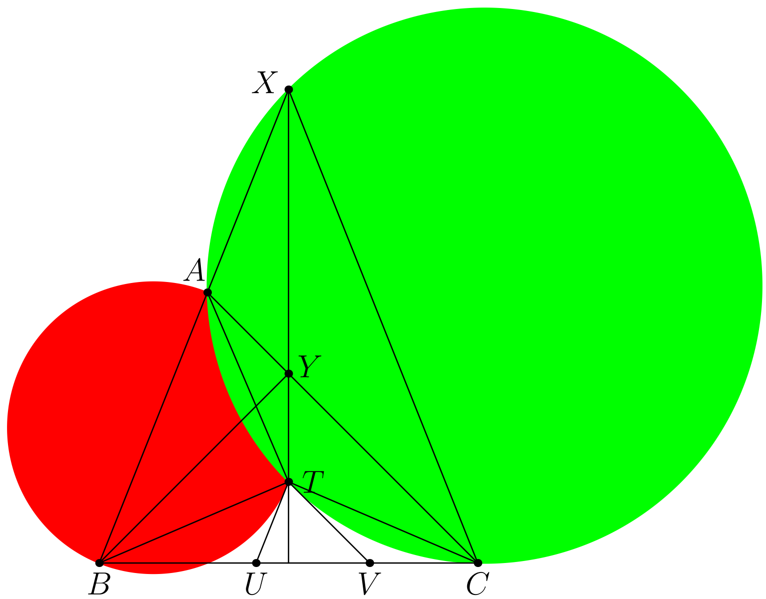
Thus,

$$\sum_{s=0}^{12} \sum_{a+b+c=s} \frac{12!\binom{4}{a}\binom{4}{b}\binom{4}{c}}{(4!)^3\binom{12}{s}} = \sum_{s=0}^{12} \frac{12!}{(4!)^3} = \frac{13!}{24^3} = \boxed{450450}.$$

Problem 13

In $\triangle ABC$, let $AB = 5, BC = 7$, and $CA = 8$. Let the perpendicular bisector of BC intersect lines AB and AC at X and Y. Let the circumcircle of $\triangle ABY$ be ω_1 and the circumcircle of $\triangle ACX$ be ω_2. Let ω_1 and ω_2 intersect at a point $T \neq A$. Let the tangents to ω_1 and ω_2 at T intersect BC at U and V, respectively. Compute UV.

Solution. The main claim is that T is the circumcenter of $\triangle ABC$.



Note that $\angle AXC = \angle BXC = 180° - 2\angle B$, so $\angle ATC = 2\angle B$. Similarly, $\angle BTA = 2\angle C$. These two facts combined show that T is the circumcenter of $\triangle ABC$.

The tangent to ω_1 at T is parallel to AB as $AT = BT$, and the tangent to ω_2 at T is parallel to AC as $AT = CT$. Thus, $\triangle TUV \sim \triangle ABC$. To find the ratio of similitude, we compute the ratio of the altitudes from T and A in these two triangles. The altitude from A to BC is

$$\frac{2[ABC]}{BC} = \frac{2 \cdot \sqrt{10 \cdot 5 \cdot 3 \cdot 2}}{7} = \frac{20\sqrt{3}}{7}.$$

Since T is the circumcenter, the altitude from T to UV is $R \cos A$. By the Law of Cosines we compute $\cos A$ as $\frac{1}{2}$, and we have

$$R = \frac{abc}{4[ABC]} = \frac{5 \cdot 7 \cdot 8}{40\sqrt{3}} = \frac{7\sqrt{3}}{3},$$

so the altitude from T to UV has length $\frac{7\sqrt{3}}{6}$. This means the ratio of similitude is

$$\frac{7\sqrt{3}/6}{20\sqrt{3}/7} = \frac{49}{120}.$$

Since $BC = 7$, this means $UV = \frac{49}{120} \cdot 7 = \boxed{\frac{343}{120}}$.

Problem 14

Compute $\displaystyle\prod_{k=1}^{8} \cos\left(\frac{k^2\pi}{17}\right)$.

Solution (Powers of 2)**.** First we compute the sign of the expression. It is easy to see that

$$\cos\left(\frac{9\pi}{17}\right), \cos\left(\frac{16\pi}{17}\right), \cos\left(\frac{25\pi}{17}\right), \cos\left(\frac{49\pi}{17}\right)$$

are the only positive terms, so the expression is positive. We now compute the absolute value of the expression.
Note that 2 is a quadratic residue mod 17 (by, for example, 36) and $\text{ord}_{17}(2) = 8$. Thus, the powers of 2 are the same as the perfect squares mod 17. Thus, we see

$$\left|\prod_{k=1}^{8} \cos\left(\frac{k^2\pi}{17}\right)\right| = \prod_{k=1}^{8} \left|\cos\left(\frac{k^2\pi}{17}\right)\right| = \prod_{k=0}^{7} \left|\cos\left(\frac{2^k\pi}{17}\right)\right| = \left|\prod_{k=0}^{7} \cos\left(\frac{2^k\pi}{17}\right)\right|.$$

We now ignore the absolute value (it turns out to be negative.) Multiplying the expression by $\sin\left(\frac{\pi}{17}\right)$, it will telescope by the identity $\sin 2\theta = 2\sin\theta\cos\theta$. Then it is equal to

$$\frac{1}{256} \cdot \frac{\sin\left(\frac{256\pi}{17}\right)}{\sin\left(\frac{\pi}{17}\right)} = -\frac{1}{256},$$

so the answer is $\boxed{\frac{1}{256}}$.

Solution (Multiplying by sin)**.** We see similar to the first solution that the answer is positive, so now we can take some absolute values.
Let $A = \displaystyle\prod_{k=1}^{8} \cos\left(\frac{k^2\pi}{17}\right)$ and $B = \displaystyle\prod_{k=1}^{8} \sin\left(\frac{k^2\pi}{17}\right)$. By the double angle formula, we have

$$|AB| = \frac{1}{2^8} \prod_{k=1}^{8} \left|\sin\left(\frac{2k^2\pi}{17}\right)\right|.$$

However, note that 2 is a quadratic residue mod 17, so the set of $2k^2$ is equal to the set of quadratic residues mod 17. Thus, we have

$$|AB| = \frac{1}{2^8}|B| \implies |A| = \frac{1}{256},$$

so $A = \boxed{\frac{1}{256}}$.

Problem 15

Let S be the set of positive integer divisors of 1000. How many functions $f : S \longrightarrow S$ are there such that for any $x, y \in S$,

$$\gcd((f(x), f(y)) = f(\gcd(x, y))?$$

Solution. We put the 16 divisors in the grid such that if we give each points coordinates such as the ones in an (x, y) plane (with the bottom left receiving $(0, 0)$) then (i, j) corresponds to $2^i 5^j$. Let $f(2^a 5^b) = 2^{x_{ab}} 5^{y_{ab}}$ for all $0 \le a, b \le 3$.

(x_{03}, y_{03})	(x_{13}, y_{13})	(x_{23}, y_{23})	(x_{33}, y_{33})
(x_{02}, y_{02})	(x_{12}, y_{12})	(x_{22}, y_{22})	(x_{32}, y_{32})
(x_{01}, y_{01})	(x_{11}, y_{11})	(x_{21}, y_{21})	(x_{31}, y_{31})
(x_{00}, y_{00})	(x_{10}, y_{10})	(x_{20}, y_{20})	(x_{30}, y_{30})

The condition now becomes

$$(\min\{x_{ab}, x_{cd}\}, \min\{y_{ab}, y_{cd}\}) = (x_{\min\{a,c\},\min\{b,d\}}, y_{\min\{a,c\},\min\{b,d\}})$$

for all $0 \le a, b, c, d \le 3$. It is clear that the x's and the y's act independently, so we count the number of ways to assign the x's and then square the result.

We need $\min\{x_{ab}, x_{cd}\} = x_{\min\{a,c\},\min\{b,d\}}$ for all $0 \le a, b, c, d \le 3$. We claim it is enough to determine the green squares. If we were to determine the green squares, then note that $\min\{x_{3i}, x_{33}\} = x_{3i}$, so $x_{3i} \le x_{33}$, and similarly $x_{j3} \le x_{33}$. Now note

$$\min\{x_{i3}, x_{3j}\} = x_{\min\{i,3\},\min\{3,j\}} = x_{ij},$$

so determining the green squares does determine the other squares. We are still left to show that this configuration does indeed work. This is not too hard to prove (but annoying to write) so it's left as an exercise to the reader.

We must have $x_{03} \le x_{13} \le x_{23} \le x_{33} \ge x_{32} \ge x_{31} \ge x_{30}$. If $x_{33} = k$, then by standard methods (such as stars and bars) there are $\binom{3+k}{3}$ solutions to one side of the inequality, so there are $\binom{3+k}{3}^2$ solutions total. Then there are

$$\binom{3}{3}^2 + \binom{4}{3}^2 + \binom{5}{3}^2 + \binom{6}{3}^2 = 1 + 16 + 100 + 400 = 517.$$

Since we have ignored the y's thus far, the answer is

$$517^2 = \boxed{267289}$$

Problem 16

How many ways are there to choose integers x, y, z between 1 and 103, inclusive, such that $x^2 + y^2 + z^2 - xyz \equiv 4 \pmod{103}$?

Solution. Treating the congruence $x^2 + y^2 + z^2 - xyz - 4 \equiv 0 \pmod{103}$ as a quadratic equation in x, we can solve it using the quadratic formula:

$$x = \frac{yz \pm \sqrt{(yz)^2 - 4y^2 - 4z^2 + 16}}{2}.$$

If y and z are fixed constants, then we can consider the discriminant Δ of this quadratic equation. If the discriminant is a nonzero perfect square $\pmod{103}$, then there are two solutions for $x \pmod{103}$. If the discriminant is a multiple of p, then there is exactly 1 solution for $x \pmod{103}$. Finally, if the discriminant is not a perfect square $\pmod{103}$, then there are no solutions for $x \pmod{103}$. For this reason, it's enough to find the number of pairs $(y, z) \pmod{103}$ such that the discriminant is:

- A nonzero perfect square $\pmod{103}$,
- A multiple of 103, or
- Not a perfect square $\pmod{103}$.

To do this, we require the *Legendre symbol* and some algebra. Because $1 + \left(\frac{\Delta}{103}\right)$ gives the number of solutions for $x \pmod{103}$ for a given pair $(y, z) \pmod{103}$, it is enough to compute

$$\sum_{y=1}^{103}\sum_{z=1}^{103}\left(1 + \left(\frac{y^2z^2 - 4y^2 - 4z^2 + 16}{103}\right)\right) = 103^2 + \sum_{y=1}^{103}\sum_{z=1}^{103}\left(\frac{y^2z^2 - 4y^2 - 4z^2 + 16}{103}\right).$$

Notice that $y^2z^2 - 4y^2 - 4z^2 + 16$ factors as $(y^2 - 4)(z^2 - 4)$, so this sum can be rewritten as

$$\begin{aligned} 103^2 + \sum_{y=1}^{103}\sum_{z=1}^{103}\left(\frac{y^2 - 4}{103}\right)\left(\frac{z^2 - 4}{103}\right) &= 103^2 + \sum_{y=1}^{103}\left(\frac{y^2 - 4}{103}\right)\sum_{z=1}^{103}\left(\frac{z^2 - 4}{103}\right) \\ &= 103^2 + \left(\sum_{y=1}^{103}\left(\frac{y^2 - 4}{103}\right)\right)\left(\sum_{z=1}^{103}\left(\frac{z^2 - 4}{103}\right)\right) \\ &= 103^2 + \left(\sum_{y=1}^{103}\left(\frac{y^2 - 4}{103}\right)\right)^2. \end{aligned}$$

Thus, it remains to compute $\sum_{y=1}^{103}\left(\frac{y^2-4}{103}\right)$. To do this, notice that

$$\sum_{y=1}^{103}\left(\frac{y^2 - 4}{103}\right) = 103 + \sum_{y=1}^{103}\left(\frac{y^2 - 4}{103}\right) - 103 = \sum_{y=1}^{103}\left(1 + \left(\frac{y^2 - 4}{103}\right)\right) - 103.$$

The sum on the right is the number of solutions to the congruence $y^2 - 4 \equiv w^2 \pmod{103}$, which we can rewrite as $y^2 - w^2 \equiv 4 \pmod{103}$. Setting $y + w = k$ and $y - w = \frac{4}{k}$ where k is nonzero $\pmod{103}$, we get a unique ordered pair (w, y) satisfying this congruence. For each nonzero k, it is possible to get a unique ordered pair (w, y) in this manner. Thus, the sum on the right is 102, implying that $\sum_{y=1}^{103}\left(\frac{y^2 - 4}{103}\right) = -1$, giving a final answer of $103^2 + 1 = \boxed{10610}$.

Solution (Epic substitution)**.** Let $p = 103$. We will first prove a lemma.

Lemma. *The equation* $x^2+y^2+z^2-xyz=4$ *holds if and only if* $x=a+\frac{1}{a}, y=b+\frac{1}{b}$, *and* $z=c+\frac{1}{c}$ *for some* $a,b,c\in\mathbb{F}_{p^2}$ *such that* $abc=1$.

Proof. First, assuming that $abc=1$, we can simplify

$$\begin{aligned}
x^2+y^2+z^2-xyz &= \left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2+\left(c+\frac{1}{c}\right)^2-\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)\\
&=4+(a^2+b^2+c^2)+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-\left(\frac{bc}{a}+\frac{ac}{b}+\frac{ab}{c}\right)-\left(\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\right)\\
&=4+(a^2+b^2+c^2)+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-\left(\frac{abc}{a^2}+\frac{abc}{b^2}+\frac{abc}{c^2}\right)-\left(\frac{a^2}{abc}+\frac{b^2}{abc}+\frac{c^2}{abc}\right)\\
&=4,
\end{aligned}$$

because $abc=1$. To prove the converse, suppose $x=r+\frac{1}{r}$ and $y=s+\frac{1}{s}$. Through the quadratic formula, we can solve for z as

$$\begin{aligned}
z&=\frac{xy\pm\sqrt{(x^2-4)(y^2-4)}}{2}\\
&=\frac{\left(r+\frac{1}{r}\right)\left(s+\frac{1}{s}\right)\pm\left(r-\frac{1}{r}\right)\left(s-\frac{1}{s}\right)}{2}
\end{aligned}$$

so either $z=rs+\frac{1}{rs}$ or $z=\frac{r}{s}+\frac{s}{r}$. If $z=rs+\frac{1}{rs}$, then we can take $a=r, b=s$, and $c=\frac{1}{rs}$, while if $z=\frac{r}{s}+\frac{s}{r}$ we can take $a=r, b=\frac{1}{s}$, and $c=\frac{s}{r}$. This proves the lemma. □

Using this lemma, we can assume that $x=a+\frac{1}{a}, y=b+\frac{1}{b}$, and $z=c+\frac{1}{c}$ where $abc=1$. Since $x,y,z\in\mathbb{F}_p$, we first find the number of triples (a,b,c) with $a,b,c\in\mathbb{F}_{p^2}$ such that $abc=1$ and $a+\frac{1}{a}, b+\frac{1}{b}$, and $c+\frac{1}{c}$ are all in $\mathbb{F}_p$. To do this, observe that the equation $x=t+\frac{1}{t}$ only has solutions a and $\frac{1}{a}$ for t. On the other hand, raising the equation $x=a+\frac{1}{a}$ to the p^{th} power gives $x^p=a^p+\frac{1}{a^p}$. Because $x^p=x$ by Fermat's Little Theorem, we see that a^p is a solution to the equation $x=t+\frac{1}{t}$ as well. In other words, either $a^p=a$ or $a^p=\frac{1}{a}$. Through similar reasoning for y and z, we arrive at the conditions

$$\begin{aligned}
a^{p+1}=1 &\text{ or } a^{p-1}=1,\\
b^{p+1}=1 &\text{ or } b^{p-1}=1,\\
c^{p+1}=1 &\text{ or } c^{p-1}=1,\\
abc&=1.
\end{aligned}$$

Lemma. *One of the two equations* $a^{p+1}=b^{p+1}=c^{p+1}=1$ *or* $a^{p-1}=b^{p-1}=c^{p-1}=1$ *must hold.*

Proof. Without loss of generality, we may assume that $a^k=b^k=1$, for some $k\in\{p-1,p+1\}$. Since $c=\frac{1}{ab}$, we see that $c^k=\frac{1}{a^kb^k}=1$, as desired. □

We now split the counting solutions into two cases, depending on the value of $k\in\{p-1,p+1\}$. It is enough to choose a and b, because $c=\frac{1}{ab}$. If $k=p-1$, then there are $(p-1)^2$ ways to choose a and b because the polynomial $t^{p-1}-1$ has exactly $p-1$ roots in $\mathbb{F}_p$. Similarly, if $k=p+1$, then there are $(p+1)^2$ ways to choose a and b because the polynomial $t^{p+1}-1$ has exactly $p+1$ roots in $\mathbb{F}_{p^2}$. In total, this gives $(p-1)^2+(p+1)^2=2p^2+2$ ways to choose a working triple (a,b,c). However, observe that if (a,b,c) gives a valid triple (x,y,z), then so does $\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$! Furthermore, the triples $(1,1,1),(1,-1,-1),(-1,1,-1),(-1,-1,1)$ were also counted twice because they satisfy both $a^{p+1}=b^{p+1}=c^{p+1}=1$ and $a^{p-1}=b^{p-1}=c^{p-1}=1$. Thus, every triple (x,y,z) was counted exactly twice, so we divide by 2 to get $p^2+1=\boxed{10610}$.