v_p and LTE

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1 Definitions

For a prime number p, let $v_p(n)$ be defined as the largest integer k such that $p^k|n$.

Definition check (please you haven't forgotten): Calculate the following:

- $v_2(49620)$
- $v_3(49620)$
- $v_7(50^7 - 1)$

In general if n is easy to write out entirely (like less than 8 digits) then to compute $v_p(n)$, dividing is a pretty good strategy. However, for the last bullet point, the question would have been pretty annoying to solve with just division.

2 First theorems

Show that

- $v_p(mn) = v_p(m) + v_p(n)$
- $v_p(m + n) \geq \min(v_p(m), v_p(n))$

Suppose $v_p(a) = m$ and Suppose $v_p(b) = n$. What is

- $v_p(\gcd(a, b))$
- $v_p(\text{lcm}(a, b))$

Use this to show that $\gcd(a, b)\,\text{lcm}(a, b) = ab$

Harder problem: Show if $a|b^2|a^3|b^4...$ then $a = b$

3 Lifting the Exponent

- For odd p, and x, y such that $p|(x - y)$ $v_p(x^n - y^n) = v_p(x - y) + v_p(n)$
- For odd p, and n odd, $v_p(x^n + y^n) = v_p(x + y) + v_p(n)$
- For $p = 2$, and x, y such that $2|(x - y)$, $v_2(x^n - y^n) = v_2(x - y) + v_2(x + y) + v_2(n) - 1$

Proofs in class

4 Problems

1. Let n be the least positive integer for which $149^n - 2^n$ is divisible by $3^3 \cdot 5^5 \cdot 7^7$. Find the number of positive integer divisors of n.
2. For a given k, find all n such that 5^k divides $2^n - 1$
3. $a^{a-1} - 1$ is never squarefree for $a > 2$
4. (ISL 1991) Find the largest k such that $1991^k | 1990^{1991^{1992}} + 1992^{1991^{1990}}$