NYC Math Team: October Test

October 23, 2020

Contest

Problem 1. How many permutations of N, Y, C, M, T are there such that there is exactly one letter between N and Y?

Problem 2. If x and y are real numbers such that $x^3 = \frac{16}{y^2}$ and $y^3 = \frac{8}{x^2}$, then compute $(x+y)^5$.

Problem 3. Find the largest positive integer n such that n^3 leaves a remainder of 1 when divided by $n - 3$.

Problem 4. In $\triangle ABC$, $AB = 13, BC = 14$, and $CA = 15$. Let D be the foot of the altitude from A to BC. Let E and F be points on segments AB and AC such that $AEDF$ is a parallelogram. Compute the area of this parallelogram.

Problem 5. How many solutions for x in the range $(0, 2020)$ are there such that

$$\lfloor x \rfloor^2 + 2020\{x\}^2 = x^2?$$

(Here, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x, while $\{x\}$ denotes $x - \lfloor x \rfloor$.)

Problem 6. Let $f(x) = \gcd(x^2, 2x + 6)$ for all positive integers x and let k be the maximum possible value of $f(x)$. Suppose n is the fifth smallest positive integer such that $f(n) = k$. Find the ordered pair (k, n).

Problem 7. Let $\triangle O_1 XY$ be isosceles, with $O_1 X = O_1 Y$ and $\angle XO_1Y \neq 90°$. Let O_2 be the circumcenter of $\triangle O_1 XY$, and let O_3 be the circumcenter of $\triangle O_2 XY$. If $\angle XO_1Y = \angle XO_3Y = \theta$, find the sum of all possible values of θ in degrees.

Problem 8. How many integers between 1 and 1023, inclusive, are there such that when expressed in binary, every 0 is adjacent to another 0, and every 1 is adjacent to another 1? (For example, the binary integer 1100_2 has this property, but the binary integer 1010_2 does not.)

Problem 9. Call a positive integer m *ridonkulous* if for all positive integers n, the set

$$\{5040n, 5040n + 1, 5040n + 2, \ldots, 5040n + 5040\}$$

contains a multiple of m. How many ridonkulous numbers are there?

Problem 10. Let a, b, and c be the roots of $x^3 - 9x - 2020 = 0$. Compute

$$(a^2 - bc)(b^2 - ca)(c^2 - ab).$$

Problem 11. In $\triangle ABC$, the midpoints of AB and AC are M and N respectively. Let BN and CM intersect at G. If $AMGN$ is cyclic, $BC = 6$, and $\angle BAC = 30°$, compute $(AB + AC)^2$.

Problem 12. Compute

$$\sum_{a=0}^{4}\sum_{b=0}^{4}\sum_{c=0}^{4} \frac{(a+b+c)!}{a!b!c!} \cdot \frac{(12 - a - b - c)!}{(4-a)!(4-b)!(4-c)!}.$$