# NYC Math Team: October Test

October 23, 2020

## Contest

**Problem 1.** How many permutations of $N, Y, C, M, T$ are there such that there is exactly one letter between $N$ and $Y$?

**Problem 2.** If $x$ and $y$ are real numbers such that $x^3 = \frac{16}{y^2}$ and $y^3 = \frac{8}{x^2}$, then compute $(x + y)^5$.

**Problem 3.** Find the largest positive integer $n$ such that $n^3$ leaves a remainder of 1 when divided by $n - 3$.

**Problem 4.** In $\triangle ABC$, $AB = 13, BC = 14$, and $CA = 15$. Let $D$ be the foot of the altitude from $A$ to $BC$. Let $E$ and $F$ be points on segments $AB$ and $AC$ such that $AEDF$ is a parallelogram. Compute the area of this parallelogram.

**Problem 5.** How many solutions for $x$ in the range $(0, 2020)$ are there such that

$$\lfloor x \rfloor^2 + 2020\{x\}^2 = x^2?$$

(Here, $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x$, while $\{x\}$ denotes $x - \lfloor x \rfloor$.)

**Problem 6.** Let $f(x) = \gcd(x^2, 2x + 6)$ for all positive integers $x$ and let $k$ be the maximum possible value of $f(x)$. Suppose $n$ is the fifth smallest positive integer such that $f(n) = k$. Find the ordered pair $(k, n)$.

**Problem 7.** Let $\triangle O_1XY$ be isosceles, with $O_1X = O_1Y$ and $\angle XO_1Y \neq 90°$. Let $O_2$ be the circumcenter of $\triangle O_1XY$, and let $O_3$ be the circumcenter of $\triangle O_2XY$. If $\angle XO_1Y = \angle XO_3Y = \theta$, find the sum of all possible values of $\theta$ in degrees.

**Problem 8.** How many integers between 1 and 1023, inclusive, are there such that when expressed in binary, every 0 is adjacent to another 0, and every 1 is adjacent to another 1? (For example, the binary integer $1100_2$ has this property, but the binary integer $1010_2$ does not.)

**Problem 9.** Call a positive integer $m$ *ridonkulous* if for all positive integers $n$, the set

$$\{5040n, 5040n + 1, 5040n + 2, \ldots, 5040n + 5040\}$$

contains a multiple of $m$. How many ridonkulous numbers are there?

**Problem 10.** Let $a, b$, and $c$ be the roots of $x^3 - 9x - 2020 = 0$. Compute

$$(a^2 - bc)(b^2 - ca)(c^2 - ab).$$

**Problem 11.** In $\triangle ABC$, the midpoints of $AB$ and $AC$ are $M$ and $N$ respectively. Let $BN$ and $CM$ intersect at $G$. If $AMGN$ is cyclic, $BC = 6$, and $\angle BAC = 30°$, compute $(AB + AC)^2$.

**Problem 12.** Compute

$$\sum_{a=0}^{4}\sum_{b=0}^{4}\sum_{c=0}^{4}\frac{(a+b+c)!}{a!b!c!}\cdot\frac{(12-a-b-c)!}{(4-a)!(4-b)!(4-c)!}.$$

**Problem 13.** In $\triangle ABC$, $AB=5, BC=7$, and $CA=8$. Let the perpendicular bisector of $BC$ intersect lines $AB$ and $AC$ at $X$ and $Y$. Let the circumcircle of $\triangle ABY$ be $\omega_1$ and the circumcircle of $\triangle ACX$ be $\omega_2$. Let $\omega_1$ and $\omega_2$ intersect at a point $T\neq A$. Let the tangents to $\omega_1$ and $\omega_2$ at $T$ intersect $BC$ at $U$ and $V$, respectively. Compute $UV$.

**Problem 14.** Compute $\displaystyle\prod_{k=1}^{8}\cos\left(\frac{k^2\pi}{17}\right)$.

**Problem 15.** Let $S$ be the set of positive integer divisors of 1000. How many functions $f: S\longrightarrow S$ are there such that for any $x, y\in S$,

$$\gcd((f(x),f(y))=f(\gcd(x,y))?$$

**Problem 16.** How many ways are there to choose integers $x, y, z$ between 1 and 103, inclusive, such that $x^2+y^2+z^2-xyz\equiv 4 \pmod{103}$?