Inversion Problems

Ethan Joo

November 6, 2020

Problems

1. (ELMO 2018 P3) Let A be a point in the plane, and ℓ a line not passing through A. Evan doesn't have a straightedge, but instead has a special compass which has the ability to draw a circle through three distinct noncollinear points. (The center of the circle is not marked in this process.) Additionally, Evan can mark the intersections between two objects drawn, and can mark an arbitrary point on a given object or on the plane.
(i) Can Evan construct the reflection of A over ℓ?
(ii) Can Evan construct the foot of the altitude from A to ℓ?

2. Let the incircle of triangle ABC touch sides BC, CA, AB at points D, E, F respectively. Let I and O be the incenter and circumcenter of triangle ABC respectively. Prove that the orthocenter of triangle DEF lies on line IO.

3. (Mixtillinear Incircles) Let Γ be the circumcircle of triangle ABC. Circle ω is tangent to AB, AC, and internally tangent to Γ at point T. Let D be the touchpoint of the A-excircle with BC. Prove that $\angle BAD = \angle TAC$.

4. (GOTEEM 2019 P2) Let ABC be an acute triangle with $AB \neq AC$, and let D, E, F be the feet of the altitudes from A, B, C, respectively. Let P be a point on DE such that $AP \perp AB$ and let Q be a point on DF such that $AQ \perp AC$. Lines PQ and BC intersect at T. If M is the midpoint of $\overline{BC}$, prove that $\angle MAT = 90°$.

5. (ISL 2014 G4) Consider a fixed circle Γ with three fixed points A, B, and C on it. Also, let us fix a real number $\lambda \in (0, 1)$. For a variable point $P \notin \{A, B, C\}$ on Γ, let M be the point on the segment CP such that $CM = \lambda \cdot CP$. Let Q be the second point of intersection of the circumcircles of the triangles AMP and BMC. Prove that as P varies, the point Q lies on a fixed circle.

6. (RMM 2009) Given four points A_1, A_2, A_3, A_4 in the plane, no three collinear, such that

$$A_1A_2 \cdot A_3A_4 = A_1A_3 \cdot A_2A_4 = A_1A_4 \cdot A_2A_3,$$

denote by O_i the circumcenter of $\triangle A_jA_kA_l$ with $\{i,j,k,l\} = \{1,2,3,4\}$. Assuming $\forall i A_i \neq O_i$, prove that the four lines A_iO_i are concurrent or parallel.

7. (China TST 2015) Triangle ABC is isoceles with $AB = AC > BC$. Let D be a point in its interior such that $DA = DB + DC$. Suppose that the perpendicular bisector of segment AB meets the external angle bisector of angle $\angle ADB$ at P, and let Q be the intersection of the perpendicular bisector of segment AC and the external angle bisector of angle $\angle ADC$. Prove that points B, C, P, Q are concyclic.

8. (USAMO 2008) Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.