Divisibility and Bases Lesson

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1 Divisibility

1.1 Divisibility Rules

What are the divisibility rules for:

- 2?
- 3?
- 4?
- 5?
- 6? How can this be applied to other numbers?
- 8?
- 9?
- 10?
- 11?

1.2 Exercises

1. Is 1748391 divisible by 9?
2. Is 3953851 divisible by 11?

1.3 Proving Divisibility Rules

- Prove the divisibility rule for 2.
- Prove the divisibility rule for 4.
- Prove the divisibility rule for 8.
- Prove the divisibility rule for 3.
- Prove the divisibility rule for 9.
- Prove the divisibility rule for 11.

1.4 Problems

1. Find the largest 3-digit number that is divisible by 22 such that the sum of the units digit and the tens digit is 11.
2. If the five-digit number $ABCDE$ is the 4th power of a whole number and $A+C+E=B+D$, find C.

2 Bases

Base: The number of digits in a number system
Base 10 is what's typically used, which has 10 digits: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

A number $A_nA_{n-1}...A_2A_1A_0$ in base 10 has a value $A_n * 10^n + A_{n-1} * 10^{n-1} + ... + A_2 * 10^2 + A_1 * 10^1 + A_0$.

Likewise, a number $A_nA_{n-1}...A_2A_1A_0$ in base b has a value $A_n * b^n + A_{n-1} * b^{n-1} + ... + A_2 * b^2 + A_1 * b^1 + A_0$.

When a number N is in a certain base b, it is written as N_b.

When writing in a base system which has more than 10 digits, the additional digits will become letters. For example, the digits in base 16 are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F$

2.1 How to convert a number c from base a to base 10

- Consider a number $c = A_nA_{n-1}...A_2A_1A_0$ in base a.
- Write it as $c = A_n * a^n + A_{n-1} * a^{n-1} + ... + A_2 * a^2 + A_1 * a^1 + A_0$.
- Multiply and add (using base 10 rules) until you simplify it down to one number. That number is c in base 10.

2.2 How to convert a number c from base 10 to base b

- Find the largest power of b that is smaller than your number c. Call that power b^n.
- Subtract b^n from c until you get a value less than b^n. Put the number of times you subtracted b^n in the b^n's place.
- Repeat for $b^{n-1}, b^{n-2}, ...b^1, 1$, putting the number of times you subtracted in the correct place.

2.3 How to convert a number c from base a to base b

To do this, you can convert the number from base a to base 10, and then from base 10 to base b, using the two sections above.

2.4 Exercises

1. Convert 83_{10} to base 8.
2. Convert 100_{10} to base 2.
3. Convert 225_7 to base 9.
4. Convert DF_{16} to base 6.

2.5 Problems

1. If N is the base 4 equivalent of 361_{10}, find the square root of N in base 4.
2. Find all solutions to the equation $72_b = 2(27_b)$ or find that there is no solution.
3. Find all solutions to the equation $73_b = 2(37_b)$ or find that there is no solution.
4. ABC and CBA are, respectively, the base 9 and base 7 numerals for the same positive integers. Express this integer in base 10.
5. The three-digit base x numeral $7y3$ is twice the three-digit base x numeral $3y7$. Express $x + y$ as a base 10 numeral.

3 Operations in Different Bases

3.1 Addition and Subtraction

Addition and subtraction are the same, except instead of carrying multiples of 10, you carry multiples of b.

3.2 Problems

1. $101101101_2 + 101000010_2$ (in base 2)
2. $DEF_{16} + ABC_{16}$ (in base 16)
3. $1A4C_{16} - DFE_{16}$ (in base 16)

3.3 Multiplication

When using the multiplication algorithm:

- In the first step, when multiplying digits, find the result in base b, then drop down the one's digit and carry over the "b's digit".
- In the second step, the addition is done as mentioned earlier.

3.4 Problems

1. $9_{19} * 9_{19}$ (in base 19)
2. $10111_2 * 1010_2$ (in base 2)
3. $57_8 * 67_8$ (in base 8)

4 Decimals in Different Bases

4.1 Converting from Base 10 to Base b (non-repeating decimals)

Way 1

- Find the largest power of b that is smaller than your number c. Call that power b^n.
- Subtract b^n from c until you get a value less than b^n. Put the number of times you subtracted b^n in the b^n's place.
- Repeat for $b^{n-1}, b^{n-2}, ...b^1, 1, b^{-1}, b^{-2}, ...$, putting the number of times you subtracted in the correct place. Stop when some subtraction gives you a result of 0.

Way 2

- Multiply c by b repeatedly until you get a whole number. Call the number of times you have multiplied k, and your new number d. Now you know $c = \frac{d}{b^k}$. You can start by converting d to base b.
- Find the largest power of b that is smaller than your number d. Call that power b^n.
- Subtract b^n from d until you get a value less than b^n. Put the number of times you subtracted b^n in the b^n's place.
- Repeat for $b^{n-1}, b^{n-2}, ...b^1, 1$, putting the number of times you subtracted in the correct place.
- Now, divide d by b^k to get your result c by shifting the decimal place to the left k times.

Note: These methods will only work if c is a non-repeating decimal in base b.

An extreme example: $\frac{1}{49} = 0.\overline{020408163265306122448979591836734693877551}_{10} = 0.01_7$.

4.2 Problems

1. Express 9.25_{10} in base 2.

2. Express 3.65625_{10} in base 2.

3. Express $11.\bar{5}_{10}$ in base 3.

5 AMC/AIME Problems

1. The base-ten representation for 19! is $121,6T5,100,40M,832,H00$, where T, M, and H denote digits that are not given. What is $T+M+H$? (2019 AMC 10B) (A=3, B=8, C=12, D=14, E=17)

2. The number n can be written in base 14 as $\underline{abc}$, can be written in base 15 as $\underline{acb}$, and can be written in base 6 as $\underline{acac}$, where $a > 0$. Find the base-10 representation of n. (2018 AIME I)

3. A positive integer N has base-eleven representation $\underline{a}\,\underline{b}\,\underline{c}$ and base-eight representation $\underline{1}\,\underline{b}\,\underline{c}\,\underline{a}$, where a, b, and c represent (not necessarily distinct) digits. Find the least such N expressed in base ten. (2020 AIME I)