Trigonometry Solutions

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Problems and Solutions

Problem 1

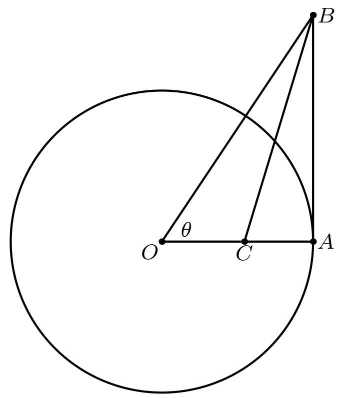
What is
$$\frac{\sin(1)\sin(2)\cdots\sin(89)}{\cos(1)\cos(2)\cdots\cos(89)}?$$
(All values are in degrees)

Solution

Let's rewrite this number in a different way:
$$\frac{\sin(1)\sin(2)\cdots\sin(89)}{\cos(1)\cos(2)\cdots\cos(89)} = \frac{\sin(1)\sin(2)\cdots\sin(89)}{\cos(89)\cos(88)\cdots\cos(1)},$$
by flipping the order we multiply in the denominator. But $\cos(x) = \sin(90 - x)$! So, $\cos(89) = \sin(1), \cos(88) = \sin(2)$, and so on. The number we want to calculate becomes
$$\frac{\sin(1)\sin(2)\cdots\sin(89)}{\sin(1)\sin(2)\cdots\sin(89)},$$
which is just $\boxed{1}$.

Problem 2 (AMC 12)

A circle centered at O has radius 1 and contains the point A. The segment AB is tangent to the circle at A and $\angle AOB = \theta$. If point C lies on $\overline{OA}$ and $\overline{BC}$ bisects $\angle ABO$, then what is OC? Your answer should be expressed in terms of trig functions.



Solution

Let's use trig functions to express all the lengths we can. We get that $\cos\theta = \frac{OA}{OB}$ and $\sin\theta = \frac{AB}{OB}$, and since $OA = 1$, we know that $OB = \sec\theta$ and $AB = \tan\theta$. Since BC is an angle bisector of $\angle OBA$, we can use the Angle Bisector theorem, which gives

$$\frac{OB}{OC} = \frac{AB}{AC},$$

or

$$\frac{\sec\theta}{OC} = \frac{\tan\theta}{AC}.$$

If $OC = x$, then $AC = 1 - x$ because $OA = 1$. Simplifying our equation, we get that

$$\frac{1}{x} = \frac{\sin\theta}{1-x}.$$

We can solve this equation for x to get

$$x = \boxed{\frac{1}{1+\sin\theta}}.$$

Problem 3 (AMC 12)

Which of the following describes the largest subset of values of y within the closed interval $[0, \pi]$ for which

$$\sin(x + y) \leq \sin(x) + \sin(y)$$

for every x between 0 and π, inclusive?

(A) $y = 0$ **(B)** $0 \leq y \leq \frac{\pi}{4}$ **(C)** $0 \leq y \leq \frac{\pi}{2}$ **(D)** $0 \leq y \leq \frac{3\pi}{4}$ **(E)** $0 \leq y \leq \pi$

Solution

Noticing that the left side of the inequality is $\sin(x + y)$, we use the sine angle addition formula. The left side becomes:

$$\sin(x + y) = \cos y \sin x + \cos x \sin y.$$

Because x and y are in the interval $[0, \pi]$, $\sin x$ and $\sin y$ are both nonnegative ($\cos x$ and $\cos y$ can still be negative, however). Since $\cos x$ and $\cos y$ are both at most 1, we see that $\cos y \sin x \leq \sin x$ and $\cos x \sin y \leq \sin y$. Adding these two inequalities gives that

$$\cos y \sin x + \cos x \sin y \leq \sin x + \sin y,$$

which means that

$$\sin(x + y) \leq \sin x + \sin y,$$

whenever $\boxed{0 \leq y \leq \pi}$.

Problem 4 (AMC 12)

The functions $\sin(x)$ and $\cos(x)$ are periodic with least period 2π. What is the least period of the function $\cos(\sin(x))$?

Solution

Notice that $\cos(\sin(x+\pi)) = \cos(-\sin(x))$ because $\sin(x+\pi) = -\sin(x)$. Since $\cos(-y) = \cos(y)$, we can simplify this to get $\cos(\sin(x+\pi)) = \cos(\sin(x))$. So the function $\cos(\sin(x))$ repeats every π.

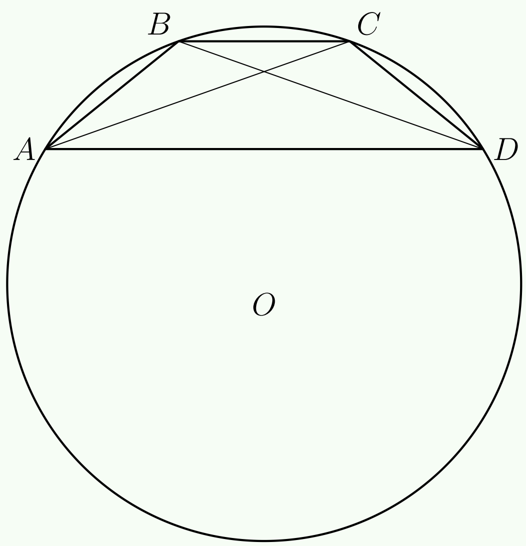
If the period is not π, then the period has to be of the form $\frac{\pi}{n}$ for some integer $n > 0$. Let the period be $p = \frac{\pi}{n} < \pi$, and notice that $\cos(\sin(0)) = \cos(0) = 1$. On the other hand, notice that $\sin(p) \neq 0$, so $\cos(\sin(p)) \neq 1$, a contradiction. This means the period of the function $\cos(\sin(x))$ is $\boxed{\pi}$.

Problem 5 (AMC 12)

A quadrilateral is inscribed in a circle of radius $200\sqrt{2}$. Three of the sides of this quadrilateral have length 200. What is the length of the fourth side?

Solution

First, let's make a diagram:



Let $\angle BAC = \theta$. Through angle chasing, we find that $ABCD$ is an isosceles trapezoid where $\angle ABD = 180 - 3\theta$. By the Law of Sines,

$$\frac{200}{\sin\theta} = \frac{AB}{\sin\theta} = 400\sqrt{2},$$

which means $\sin\theta = \frac{\sqrt{2}}{4}$. Since θ is acute, $\cos\theta = \frac{\sqrt{14}}{4}$. By the Law of Sines,

$$\frac{AD}{\sin(3\theta)} = \frac{AD}{\sin(180 - 3\theta)} = 400\sqrt{2}$$

which means we want to compute $\sin(3\theta)$. Notice that $\sin(3\theta) = \sin(2\theta)\cos\theta + \cos(2\theta)\sin\theta$ using the sine angle addition formula, and

$$\sin(2\theta) = 2 \cdot \frac{\sqrt{2}}{4} \cdot \frac{\sqrt{14}}{4} = \frac{\sqrt{7}}{4}$$

while

$$\cos(2\theta) = 1 - 2\left(\frac{\sqrt{2}}{4}\right)^2 = \frac{3}{4}.$$

Now, we can compute

$$\sin(3\theta) = \frac{\sqrt{7}}{4} \cdot \frac{\sqrt{14}}{4} + \frac{3}{4} \cdot \frac{\sqrt{2}}{4} = \frac{5\sqrt{2}}{8}.$$

This means

$$AD = \frac{5\sqrt{2} \cdot 400\sqrt{2}}{8} = \boxed{500}.$$

Problem 6 (AMC 12)

Alice and Bob live 10 miles apart. One day Alice looks due north from her house and sees an airplane. At the same time Bob looks due west from his house and sees the same airplane. The angle of elevation of the airplane is 30° from Alice's position and 60° from Bob's position. Which of the following is closest to the airplane's altitude, in miles?

(A) 3.5 **(B)** 4 **(C)** 4.5 **(D)** 5 **(E)** 5.5

Solution

Let the altitude of the airplane be x. Then, the distance from Alice to the point directly underneath the airplane is $x\sqrt{3}$ (using the trig formulas for 30°!) and the distance from Bob to the point directly underneath the airplane is $\frac{x}{\sqrt{3}}$. Since the airplane is directly north from Alice and the airplane is directly west from Bob, the triangle formed by the point underneath the plane, Bob, and Alice is a right triangle. This means the squared distance between Alice and Bob is $3x^2 + \frac{x^2}{3} = \frac{10x^2}{3}$ by the Pythagorean theorem, which is 100 because the distance between Alice and Bob is given to be 10. This means $x = \sqrt{30}$, and this is closest to $\boxed{5.5}$.

Problem 7 (AMC 12)

A bee starts flying from point P_0. She flies 1 inch due east to point P_1. For $j \geq 1$, once the bee reaches point P_j, she turns 30° counterclockwise and then flies $j+1$ inches straight to point P_{j+1}. When the bee reaches P_{2015} she is exactly $a\sqrt{b}+c\sqrt{d}$ inches away from P_0, where a, b, c and d are positive integers and b and d are not divisible by the square of any prime. What is $a+b+c+d$?

Solution

Let's use coordinate geometry along with trigonometry to keep track of the bee's path. In the x direction, the bee's final position is

$$x = 1\cos 0 + 2\cos 30 + 3\cos 60 + \ldots + 2014\cos 270 + 2015\cos 300.$$

To simplify this, we can use the fact that $\cos(\theta + 180) = -\cos\theta$. Grouping like terms gives:

$$\begin{aligned} x = &(1 - 7 + 13 - \cdots + 2005 - 2011)\cos 0 + (2 - 8 + 14 - \cdots + 2006 - 2012)\cos 30 + \cdots \\ &+(5 - 11 + 17 - \cdots + 2009 - 2015)\cos 120 + (6 - 12 + 18 - \cdots - 2004 + 2010)\cos 150, \end{aligned}$$

and notice that each sum has 336 terms except for the last sum. Each sum evaluates to -1008, except the last sum, which is 1008. Thus

$$x = -1008\cos 0 - 1008\cos 30 - 1008\cos 60 - 1008\cos 90 - 1008\cos 120 + 1008\cos 150,$$

so $x = -1008(1+\sqrt{3})$.
In the y direction, the bee's final position is

$$y = 1\sin 0 + 2\sin 30 + 3\sin 60 + \ldots + 2014\sin 270 + 2015\sin 300,$$

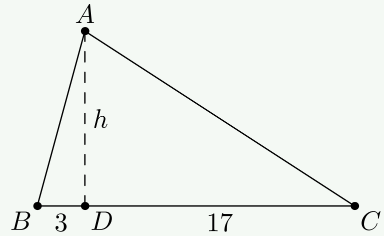
and using similar reasoning,

$$y = -1008\sin 0 - 1008\sin 30 - 1008\sin 60 - 1008\sin 90 - 1008\sin 120 + 1008\sin 150,$$

so $y = -1008(1+\sqrt{3})$ as well. The distance from P_{2015} to P_0, using the distance formula, is $\sqrt{x^2+y^2} = 1008(1+\sqrt{3})\sqrt{2} = 1008\sqrt{2} + 1008\sqrt{6}$. Then, $a+b+c+d = \boxed{2024}$.

Problem 8 (AIME)

In triangle ABC, $\tan\angle CAB = 22/7$, and the altitude from A divides BC into segments of length 3 and 17. What is the area of triangle ABC?



Solution

Using the definition of tangent, we see that $\tan\angle CAD = \frac{17}{h}$ and $\tan\angle DAB = \frac{3}{h}$. Using this information, we see that

$$\tan\angle CAB = \tan(\angle CAD + \angle DAB) = \frac{\frac{17}{h} + \frac{3}{h}}{1 - \frac{17}{h}\cdot\frac{3}{h}}$$

by the tangent angle addition formula. Simplifying gives

$$\tan(\angle CAD + \angle DAB) = \frac{20h}{h^2 - 51} = \frac{22}{7},$$

and we can solve this equation for h. Cross-multiplying gives

$$11h^2 - 70h - 561 = 0,$$

which means $h = 11$. Using the formula $K = \frac{bh}{2}$ to find the area, we get $\boxed{110}$ as the area of $\triangle ABC$.

Problem 9 (AIME)

Given that $A_k = \frac{k(k-1)}{2}\cos\frac{k(k-1)\pi}{2}$, find $|A_{19} + A_{20} + \cdots + A_{98}|$.

Solution

Notice that the angle, $\frac{k(k-1)\pi}{2}$ is always a multiple of π. This means that the cosine term is always either 1 or -1. When $\frac{k(k-1)}{2}$ is even, the cosine term is 1, and when it is odd, the cosine term is -1. Notice that $\frac{k(k-1)}{2}$ is even when k or $k-1$ is a multiple of 4, and odd otherwise. This means the sum is

$$-\frac{19(18)}{2} + \frac{20(19)}{2} + \frac{21(20)}{2} - \frac{22(21)}{2} - \frac{23(22)}{2} + \frac{24(23)}{2} \cdots - \frac{98(97)}{2}.$$

Notice that every pair in the sum adds to $-\frac{(n)(n-1)}{2} + \frac{(n+1)(n)}{2} = n$, so the sum can be expressed as $19 - 21 + 23 ... + 95 - 97$. Each pair of consecutive terms adds to -2, and there are 20 pairs. Thus, the answer is $|-40| = \boxed{40}$.

Problem 10 (AIME)

Given that $(1+\sin t)(1+\cos t) = 5/4$ and

$$(1-\sin t)(1-\cos t) = \frac{m}{n} - \sqrt{k},$$

where k, m, and n are positive integers with m and n relatively prime, find $k+m+n$.

Solution

Let $x = (1-\sin t)(1-\cos t)$. Multiplying by $\frac{5}{4} = (1+\sin t)(1+\cos t)$, we see that $\frac{5x}{4} = (1-\sin^2 t)(1-\cos^2 t)$. Since $\sin^2 t + \cos^2 t = 1$,

$$\frac{5x}{4} = \sin^2 t \cos^2 t,$$

so $\sin t \cos t = \frac{\sqrt{5x}}{2}$. Since $(1+\sin t)(1+\cos t) = \frac{5}{4}$, we can expand this to get $\sin t + \cos t = \frac{1}{4} - \frac{\sqrt{5x}}{2}$, and squaring gives

$$1 + 2\sin t \cos t = \frac{1}{16} - \frac{\sqrt{5x}}{4} + \frac{5x}{4},$$

and simplifying this equation gives

$$0 = \frac{5x}{4} - \frac{5\sqrt{5x}}{4} - \frac{15}{16},$$

or that

$$0 = x - \sqrt{5x} - \frac{3}{4}.$$

Letting $y = \sqrt{x}$, we get that $0 = y^2 - y\sqrt{5} - \frac{3}{4}$, and we can solve this to get $y = -\sqrt{2} + \frac{\sqrt{5}}{2}$. Finally, squaring this, we get that $x = \frac{13}{4} - \sqrt{10}$, for a final answer of $\boxed{27}$.