Trigonometry and Geometry

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Trigonometry can be a very useful tool for solving geometry problems.

1 Some More Trig Identities

Many problems involve a sum of trig functions when it is actually helpful to interpret it as a product of trig functions, and vice-versa. The product-to-sum and the sum-to-product formulas allow us to tackle these types of problems.

Theorem 1 (Product to Sum Formulas)
The following formulas hold:

- $\sin x \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$.
- $\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$.
- $\cos x \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$.

Proof
Expand using the angle addition and subtraction formulas.

Using these formulas, we can also convert sums of trig functions to products of trig functions.

Theorem 2 (Sum to Product Formulas)
The following formulas hold:

- $\sin u + \sin v = 2\sin\frac{u+v}{2}\cos\frac{u-v}{2}$.
- $\cos u + \cos v = 2\cos\frac{u+v}{2}\cos\frac{u-v}{2}$.
- $\cos u - \cos v = -2\sin\frac{u+v}{2}\sin\frac{u-v}{2}$.

Proof
Starting with the Product to Sum formulas, set $u = x+y$ and $v = x-y$. Then, $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$. Then, the Sum to Product formulas hold as a result of the Product to Sum formulas.

Exercises

Exercise 1. Simplify $\sin 40 + \sin 50$ into one term using sum to product formulas.

Exercise 2. Simplify $\sin 20 \cdot \sin 40$ using product to sum formulas.

2 Law of Sines

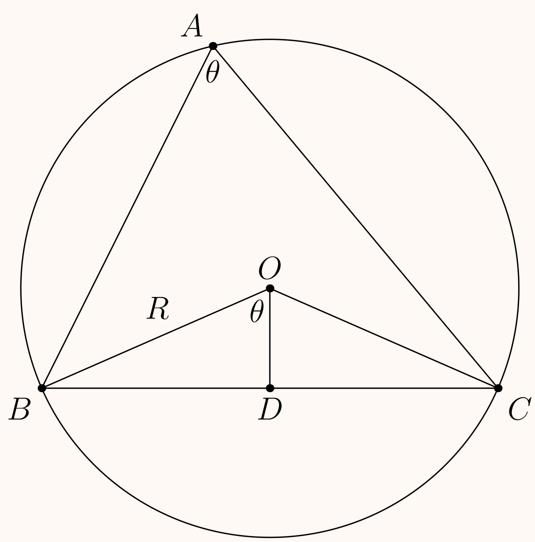
One way to relate the sides of a triangle to the angles opposite them is through the Law of Sines.

Theorem 3 (Law of Sines)
In triangle ABC,

$$\frac{BC}{\sin A} = \frac{AC}{\sin B} = \frac{AB}{\sin C} = 2R,$$

where R is the circumradius of triangle ABC.

Proof
First, a diagram:



Notice that $\angle BOC = 2\angle BAC$. Dropping a perpendicular from the center O to BC at D cuts the length of BC in half, since $\triangle BOC$ is isosceles. This perpendicular also cuts $\angle BOC$ in half, so $\angle BOD = \angle BAC$. However, notice that $\sin \angle BOD = \frac{BC}{2R}$, or that

$$\frac{BC}{\sin \angle BOD} = 2R,$$

as desired.

This proof only covers the case where $\triangle ABC$ is an acute triangle, but the Law of Sines still holds true for obtuse triangles (can you prove this?).

Exercises

Exercise 1. Prove the Law of Sines in the case where triangle ABC is obtuse.

Exercise 2. Suppose triangle ABC satisfies $AB = 5$ and $AC = 4$, and $\sin \angle ABC = \frac{1}{2}$. What is $\sin \angle ACB$?

Exercise 3. Suppose triangle ABC satisfies $AB = 5$ and $AC = 4$, and $\sin \angle ABC = \frac{1}{2}$. What is the circumradius of $\triangle ABC$?

3 Law of Cosines

Another way to relate the sides of a triangle to its angles is through the Law of Cosines.

Theorem 4 (Law of Cosines)
In triangle ABC with $BC = a$, $AC = b$, and $AB = c$, the following identities hold:

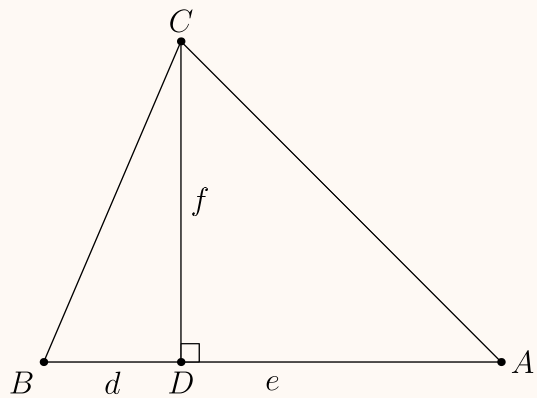
$$c^2 = a^2 + b^2 - 2ab\cos\angle C$$

$$b^2 = a^2 + c^2 - 2ac\cos\angle B$$

$$a^2 = b^2 + c^2 - 2bc\cos\angle A$$

Proof

First, a diagram:



From the Pythagorean theorem, we see that

$$a^2 + b^2 - 2f^2 = d^2 + e^2.$$

Listing out all information we have through trigonometry, we see that $\cos\angle BCD = \frac{f}{a}, \sin\angle BCD = \frac{d}{a}, \cos\angle ACD = \frac{f}{b}$, and $\sin\angle ACD = \frac{e}{b}$. Using the cosine angle addition formula, we see that

$$\cos\angle C = \cos(\angle BCD + \angle DCA) = \frac{f}{a}\cdot\frac{f}{b} - \frac{d}{a}\cdot\frac{e}{b} = \frac{f^2 - de}{ab}.$$

This means

$$-2ab\cos\angle C = 2de - 2f^2,$$

implying that

$$a^2 + b^2 - 2ab\cos\angle C = d^2 + 2de + e^2 = (d+e)^2 = c^2,$$

as desired.

Exercises

Exercise 1. In triangle ABC, $AB = 3$, $AC = 8$, and $\angle A = 60°$. What is BC?

Exercise 2. In triangle ABC, $AB = 7$, $AC = 5$, and $BC = 3$. What is $\cos\angle C$?

4 Problems

Problem 1. Triangle ABC has side lengths $AB = 13, BC = 14$, and $AC = 15$. What is the area of triangle ABC?

Problem 2 (AMC 12)**.** Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3 \cdot AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3 \cdot BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3 \cdot CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?

Problem 3 (AMC 12)**.** An object moves 8 cm in a straight line from A to B, turns at an angle α, measured in radians and chosen at random from the interval $(0, \pi)$, and moves 5 cm in a straight line to C. What is the probability that $AC < 7$?

Problem 4 (Stewart's Theorem)**.** In $\triangle ABC$, we draw cevian AD. If $AD = d, BD = m, CD = n, AB = c, AC = b$ and $BC = a$, prove that

$$amn + d^2a = b^2m + c^2n.$$

Problem 5 (AMC 12)**.** In $\triangle ABC$ with integer side lengths,

$$\cos A = \frac{11}{16}, \qquad \cos B = \frac{7}{8}, \qquad \text{and} \qquad \cos C = -\frac{1}{4}.$$

What is the least possible perimeter for $\triangle ABC$?

Problem 6 (AIME)**.** In equilateral $\triangle ABC$ let points D and E trisect $\overline{BC}$. Then $\sin(\angle DAE)$ can be expressed in the form $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is an integer that is not divisible by the square of any prime. Find $a + b + c$.

Problem 7 (Ratio Lemma)**.** In triangle ABC, we draw cevian AD. Prove that

$$\frac{BD}{CD} = \frac{AB \sin \angle BAD}{AC \sin \angle CAD}.$$

Problem 8 (HMMT)**.** Compute the value of

$$\frac{\cos 30.5 + \cos 31.5 + \ldots + \cos 44.5}{\sin 30.5 + \sin 31.5 + \ldots + \sin 44.5}.$$

Problem 9 (AMC 12)**.** Suppose that $\triangle ABC$ is an equilateral triangle of side length s, with the property that there is a unique point P inside the triangle such that $AP = 1$, $BP = \sqrt{3}$, and $CP = 2$. What is s?

Problem 10 (AIME)**.** Triangle ABC has side lengths $AB = 7, BC = 8$, and $CA = 9$. Circle ω_1 passes through B and is tangent to line AC at A. Circle ω_2 passes through C and is tangent to line AB at A. Let K be the intersection of circles ω_1 and ω_2 not equal to A. Then $AK = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.