NYCMT: Radical Axes

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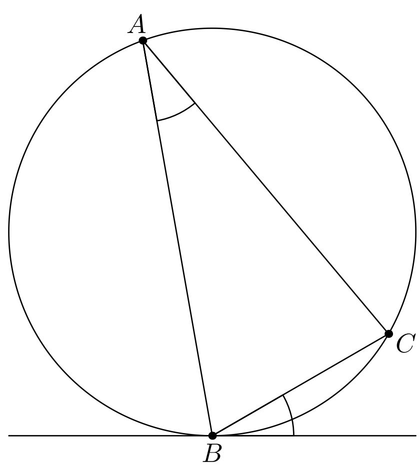
0 Power of a Point Review

- Last week, we stated that if one has a point P and a circle ω and you draw a line through P intersecting X and Y, the power of point P with respect to a circle ω was the product of the directed segments PX and PY.
- We also showed that this power is constant no matter what line you draw through P.
- Finally, we showed that the power can be numerically computed as $OP^2 - r^2$.

Also, remember the fact below; you'll see it pop up today.



1 Radical Axes

1.1 Definition

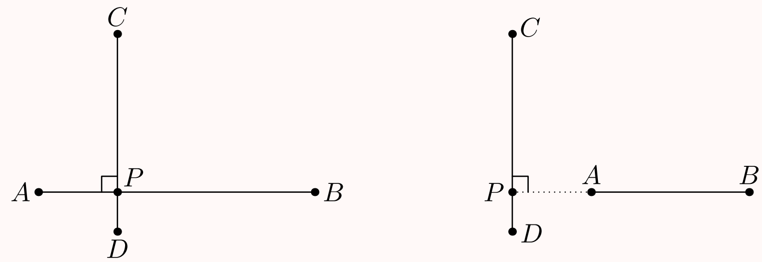
Theorem (Definition of a Radical Axis)

Suppose we have two non-concentric circles, ω and Γ. Then the *radical axis* of ω and γ is the locus of points P such that $\mathrm{Pow}(P, \omega) = \mathrm{Pow}(P, \Gamma)$.

1.2 The Radical Axis is a Line

Lemma 1 (Perpendicularity Lemma)

Given points A, B, C, D on the plane, $AB \perp CD$ if and only if $AC^2 - AD^2 = BC^2 - BD^2$.



Proof (Only If Direction)

Firstly, we show the only if direction. Suppose that line AB meets line CD at a point P.

Then, by the Pythagorean theorem, we have that $BP^2 + PC^2 = BC^2, PB^2 + PD^2 = BD^2$. Therefore, $BD^2 - BC^2 = PD^2 - PC^2$.

Likewise, $AD^2 - AC^2 = PD^2 - PC^2$. Thus, $BD^2 - BC^2 = AD^2 - AC^2$, and we are done.

Proof (If Direction)

The if direction is fairly tedious, and contains a few cases, all of which are fairly similar. We will thus only show one case; that when segments AB, CD intersect at some point P.

In this case, suppose WLOG $\angle APC = \angle BPD = \theta < 90°$. Then from the law of cosines, we have the relations

$$AC^2 = PA^2 + PC^2 - 2PA \cdot PC \cos\theta,$$

$$AD^2 = PA^2 + PD^2 + 2PA \cdot PD \cos\theta,$$

$$BC^2 = PB^2 + PC^2 + 2PB \cdot PC \cos\theta,$$

$$BD^2 = PB^2 + PD^2 - 2PB \cdot PD \cos\theta.$$

Therefore, we have that $PC^2 + PD^2 - 2PA \cdot (PC + PD)\cos\theta = AC^2 - AD^2 = BC^2 - BD^2 = PC^2 + PD^2 + 2PB \cdot (PC + PD)\cos\theta$.

This implies that $2 \cdot (PC + PD) \cdot (PA + PB) \cdot \cos\theta = 0$, which in turn implies $\cos\theta = 0$, so $\theta = 90°$, and we are done. The other cases follow in similar manner.

Now, we're ready to make the main claim of this section.

Theorem

The Radical Axis of any two non concentric circles is a line perpendicular to the line containing the centers of the circles.

Proof (Perpendicularity of the Radical Axis to the Lines Connecting the Centers)

Suppose we have two circles, ω_1 and ω_2, with centers O_1, O_2 and radii r_1, r_2. Suppose P is a point on the radical axis. Then we have

$$PO_1^2 - r_1^2 = PO_2^2 - r_2^2.$$

or

$$PO_1^2 - PO_2^2 = r_1^2 - r_2^2.$$

Now suppose Q is any other point on the radical axis. Then $QO_1^2 - QO_2^2 = r_1^2 - r_2^2 = PO_1^2 - PO_2^2$. This is true if and only if QP is perpendicular to O_1O_2. This proves the claim, since we must have that all points on the radical axis lie on a line perpendicular to O_1O_2.

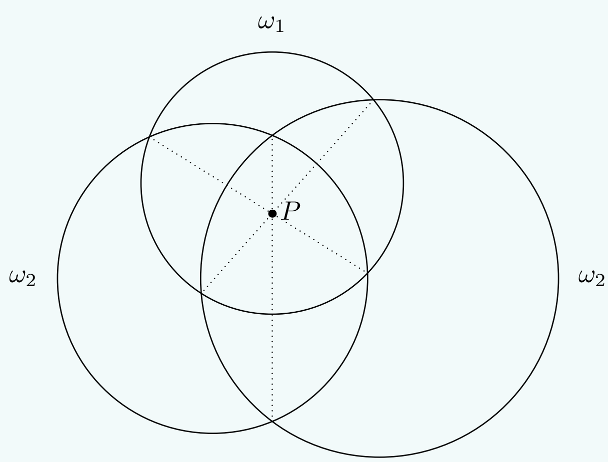
§1.2.1 Follow-Up Questions

- If two circles intersect at two points, what is their radical axis?
- If two circles are internally or externally tangent, what is their radical axis?
- (More challenging – see 2 from last week for an idea) If two circles are disjoint and neither circle is contained within the other, what is their radical axis?

1.3 The Radical Center

Theorem (Existence of the Radical Center)

Suppose we have three circles, $\omega_1, \omega_2, \omega_3$ no two of which are concentric, and whose centers are non collinear. Then, the three pairwise radical axes of these circles concur.



Proof

Since the centers are not collinear, the radical axis of ω_1 and ω_2 concurs with the radical axis of ω_1 and ω_3 at a point P. This point has equal power wrt ω_1 and ω_2, and it has equal power wrt ω_1 and ω_3, so it has equal power wrt ω_2 and ω_3, so it lies on their radical axis as well. Thus, the pairwise radical axes of these three circles are concurrent.

Example 1 (Existence of the Orthocenter)

Prove that the three altitudes in an acute triangle ABC are concurrent.

Solution. Suppose the feet of the altitudes from A, B, C are D, E, F respectively. Since $\angle BEC = \angle BFC = 90°$, B, E, F, C is cyclic. Similarly, A, B, D, E are cyclic. Finally, A, F, D, C are cyclic. The pairwise radical axes of these circles are BE, CF, and AD, the altitudes of the triangle, and therefore they concur. (At the orthocenter).

Example 2 (JMO 2012/1)

Given a triangle ABC, let P and Q be points on segments AB and AC, respectively, such that $AP = AQ$. Let S and R be distinct points on segment BC such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

By the tangent angle theorem that we established when showing the power is constant, we have that AB is tangent to the circumcircle of $\triangle PRS$, and that AC is tangent to the circumcircle of QRS.

Now, suppose that the circumcircles of PRS and QRS were distinct. Then R and S would be on their radical axis. However, since $\text{Pow}(A, (PRS)) = AP^2 = AQ^2 = \text{Pow}(A, (QRS))$, A would also lie on the radical axis. Then A, R, S would be collinear, contradiction.

§1.3.1 Follow-Up Questions

- What happens if the centers of the circles are collinear?
- How can you construct the radical axis of two circles, where one circle is fully contained within the other?

1.4 Degenerate Circles

A topic that very rarely comes up on problems using radical axis are circles with radius zero, or, as they're more commonly known, points. We will see an example using this topic shortly; typically, you treat a point as a circle of radius zero and apply radical axis. A couple of notes:

- For clear reasons, the power of any point with respect to another point is the distance between the points squared.
- Any line passing through a degenerate circle is "tangent" to that circle.

Example 3 (Bonus: Existence of the Circumcenter)

Prove that the three perpendicular bisector in an acute triangle ABC are concurrent.

Solution. Consider the degenerate circles of radius 0: points A, B, C. The radical axis of degenerate circles A and B is the perpendicular bisector of $\overline{AB}$. Similarly, the radical axis of degenerate circles A and C is the perpendicular bisector of $\overline{AB}$ and the radical axis of degenerate circles B and C is the perpendicular bisector of $\overline{BC}$. From the existence of the radical center, we have that these lines are concurrent at the circumcenter.

2 Exercises

Problem 1 (HMMT 2016)**.** Nine pairwise noncongruent circles are drawn in the plane such that any two circles intersect twice. For each pair of circles, we draw the line through these two points, for a total of $\binom{9}{2} = 36$ lines. Assume that all 36 lines drawn are distinct. What is the maximum possible number of points which lie on at least two of the drawn lines?

Problem 2 (USAMO 2009)**.** Given circles ω_1 and ω_2 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R and S lie on a circle then the center of this circle lies on line XY.

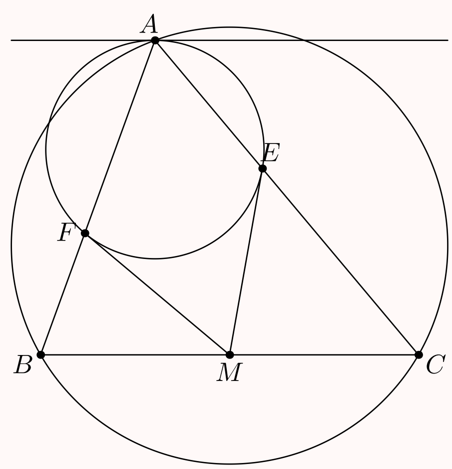
Problem 3 (IMO 1995)**.** Let A, B, C, and D be four distinct points on a line, in that order. The circles with diameters $\overline{AC}$ and $\overline{BD}$ intersect at X and Y. The line $\overline{XY}$ meets $\overline{BC}$ at Z. Let P be a point on the line $\overline{XY}$ other than Z. The line CP intersects the circle with diameter $\overline{AC}$ at C and M, and the line BP intersects the circle with diameter $\overline{BD}$ at B and N. Prove that the lines AM, DN, and XY are concurrent.

Problem 4 (USAMO 1998)**.** Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent AB to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let C be the second point of intersection of AB and $\mathcal{C}_1$, and let D be the midpoint of AB. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

Problem 5 (2016 AIME)**.** Circles ω_1 and ω_2 intersect at points X and Y. Line ℓ is tangent to ω_1 and ω_2 at A and B, respectively, with line AB closer to point X than to Y. Circle ω passes through A and B intersecting ω_1 again at $D \neq A$ and intersecting ω_2 again at $C \neq B$. The three points C, Y, D are collinear, $XC = 67$, $XY = 47$, and $XD = 37$. Find AB^2.

Lemma 2 (Three Tangents)

Suppose we have an acute triangle ABC, and E and F are the feet of the altitudes from B and C respectively. Let M be the midpoint of $\overline{BC}$. Then $\overline{ME}$ and $\overline{MF}$ are tangent to the circumcircle of $\triangle AEF$, and so is the line through A parallel to $\overline{BC}$.



This lemma is necessary for the problem below. You're not required to prove it (it's not power of a point), but we will go over it when reviewing.

Problem 6 (Iran TST)**.** In acute triangle ABC, $\angle B$ is greater than $\angle C$. Let M be the midpoint of BC and let E and F be the feet of the altitudes from B and C, respectively. Let K and L be the midpoints of ME and MF, respectively, and let T be on line KL such that $TA \parallel BC$. Prove that $TA = TM$.