Inversion Problems

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Problems

1. (ELMO 2018 P3) Let A be a point in the plane, and ℓ a line not passing through A. Evan doesn't have a straightedge, but instead has a special compass which has the ability to draw a circle through three distinct noncollinear points. (The center of the circle is not marked in this process.) Additionally, Evan can mark the intersections between two objects drawn, and can mark an arbitrary point on a given object or on the plane.
   (i) Can Evan construct the reflection of A over ℓ?
   (ii) Can Evan construct the foot of the altitude from A to ℓ?

2. Let the incircle of triangle ABC touch sides BC, CA, AB at points D, E, F respectively. Let I and O be the incenter and circumcenter of triangle ABC respectively. Prove that the orthocenter of triangle DEF lies on line IO.

3. (Mixtillinear Incircles) Let Γ be the circumcircle of triangle ABC. Circle ω is tangent to AB, AC, and internally tangent to Γ at point T. Let D be the touchpoint of the A-excircle with BC. Prove that $\angle BAD = \angle TAC$.

4. (GOTEEM 2019 P2) Let ABC be an acute triangle with $AB \neq AC$, and let D, E, F be the feet of the altitudes from A, B, C, respectively. Let P be a point on DE such that $AP \perp AB$ and let Q be a point on DF such that $AQ \perp AC$. Lines PQ and BC intersect at T. If M is the midpoint of $\overline{BC}$, prove that $\angle MAT = 90°$.