Angle Chasing

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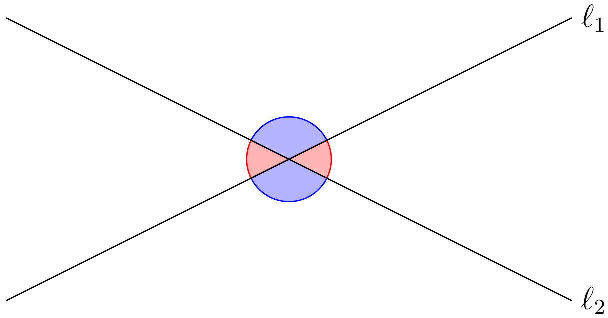
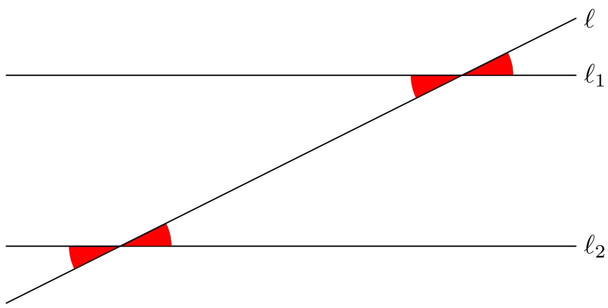
Angle chasing is the foundation of geometry, so we go in depth on the technique. Although it is built on very few rules, it can get very tricky at times.

1 Basic Facts



Fact 1

If ℓ_1 and ℓ_2 intersect at a point, then we split the 360° around this point into four angles, and opposite pairs of angles are equal to each other.



Fact 2

If $\ell_1 \parallel \ell_2$ and ℓ is a line not parallel to these two lines, then the acute angle formed by ℓ_1 and ℓ is congruent to the acute angle formed by ℓ_2 and ℓ.

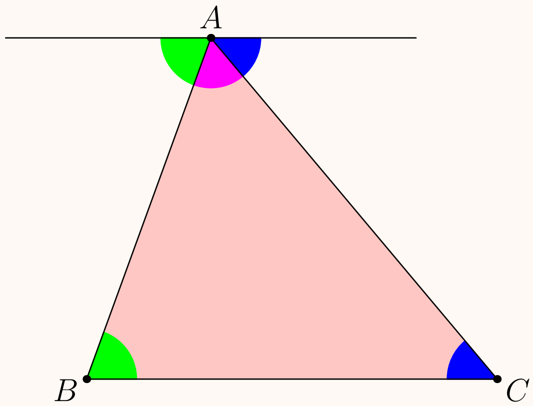
It is important to note that the converse of Fact 2 is true; that is, if ℓ intersects lines ℓ_1 and ℓ_2 and creates equal angles, we must have $\ell_1 \parallel \ell_2$.
Fact 2 is actually enough to prove one of the most imporant facts when it comes to angle chasing.

Example

Prove that the sum of the angles of any triangle ABC is $180°$.

Proof

Draw a triangle and draw the line ℓ parallel to BC through A.



By Fact 2, we see that $\angle ACB$ is equal to the angle between AC and ℓ (i.e. the blue angles are equal). Similarly, we see that $\angle CBA$ is equal to the angle between AB and ℓ (i.e. the green angles are equal). Then the sum of the angles of the triangle is equal to the angle of a line, which is just $180°$.

In fact, a much more general result is true.

Theorem (Sum of angles in a polygon with n sides)

The sum of the angles in an n-gon (a polygon with n sides) is $180°(n-2)$.

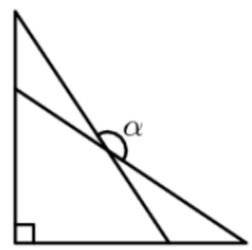
The proof of this fact is a lot harder than the above proof. If you want, try to prove this fact! (Hint: Show that you can partition any polygon into triangles.)

2 Some Exercises

Exercise 1 (2019 DMI Marathon/1)**.** In a quadrilateral, the angles form a geometric sequence with common ratio 2019. Compute the average of all the angles in the quadrilateral.

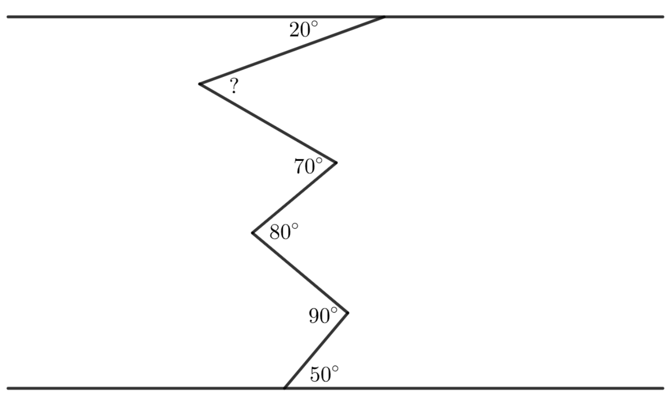
Exercise 2 (2020 AMC 10B/4)**.** The acute angles of a right triangle are $a°$ and $b°$, where $a > b$ and both a and b are prime numbers. What is the least possible value of b?

Exercise 3 (2019 CMIMC Geometry/1)**.** The figure below depicts two congruent triangles with angle measures $40°, 50°$, and $90°$. What is the measure of the obtuse angle α formed by the hypotenuses of these two triangles?



Exercise 4 (2018 CMIMC Geometry/1)**.** Let ABC be a triangle. Point P lies in the interior of $\triangle ABC$ such that $\angle ABP = 20^\circ$ and $\angle ACP = 15^\circ$. Compute $\angle BPC - \angle BAC$.

Exercise 5. In the following diagram, what is the angle labeled with "?"? (The two long lines are parallel.)

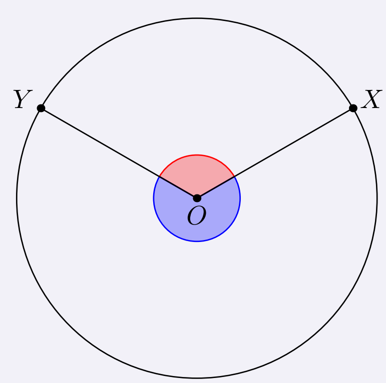


3 Circles and Angles

First we define what we mean by the measure of arc $\widehat{XY}$.

Definition

For points X and Y on a circle with center O, then the measure of arc $\widehat{XY}$ (often shortened as $\widehat{XY}$) is defined to be either $\angle XOY$ or the reflex angle $\angle XOY$, depending on context.



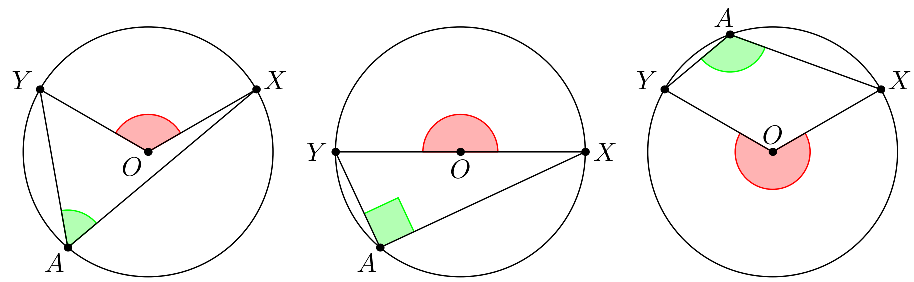
The smaller of the two arcs is called "minor arc $\widehat{XY}$", while the other one is called "major arc $\widehat{XY}$".

With this definition, we can state the inscribed angle theorem.

Theorem (Inscribed Angle Theorem)

If X, A, and Y are points on a circle centered at O, then $\angle XAY$ is equal to half of $\widehat{XY}$, where we choose either the normal angle or the reflex angle so that it and $\angle XAY$ "point in the same direction".

The statement of the theorem isn't entirely clear, so here are a few examples.



To prove the inscribed angle theorem, we require one more fact.

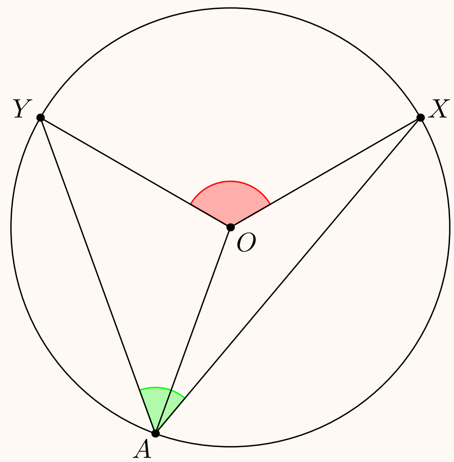
Fact 3

If $\triangle ABC$ is isosceles with $AB = AC$, then $\angle ABC = \angle ACB$.

Now we can prove the inscribed angle theorem.

Proof

There are 3 cases: when A is on minor arc XY, major arc XY, or if XY is a diameter. All three cases are similar, so we just show the first case. You should complete the other two cases though.



Label $\angle OAY = \alpha$ and $\angle OAX = \beta$. Since $OA = OY$, we have $\angle OAY = \angle OYA = \alpha$. Since the sum of the angles of $\triangle OAY$ is $180°$, we must have $\angle AOY = 180° - 2\alpha$. Similarly, $\angle AOX = 180° - 2\beta$. Then

$$\angle XOY = 360° - (\angle AOY + \angle AOX) = 360° - ((180° - 2\alpha) + (180° - 2\beta)) = 2(\alpha + \beta) = 2\angle XAY$$

as desired.

Note that this implies that if A lies on a circle with diameter XY, then $\angle XAY = 90°$.

4 More Exercises

Exercise 1. If A, B, C, D lie on a circle in that order, prove that $\angle BAC = \angle BDC$.

Exercise 2. If A, B, C, D lie on a circle in that order, prove that $\angle ABC + \angle CDA = \angle DAB + \angle BCD = 180^\circ$.

Exercise 3. Suppose A, B, C, D lie on a circle such that AC and BD intersect inside the circle at a point P. Show that $\angle APB = \frac{\widehat{AB}+\widehat{CD}}{2}$.

Exercise 4. Suppose A, B, C, D lie on a circle such that the extension of AB past B and the extension of CD past C intersect outside the circle at a point P. Show that $\angle BPC = \frac{\widehat{AD}-\widehat{BC}}{2}$

Exercise 5 (Reim's Theorem)**.** Let ω_1 and ω_2 be two circles that intersect at X and Y. Draw a line through X that intersects ω_1 at A and ω_2 at B. Draw a line through Y that intersects ω_1 at C and ω_2 at D. Prove that $AC \parallel BD$.