NYCMT: Power of a Point

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10/9

0 Review

0.1 Angles

- **Angles under the same arc**: Suppose A, B, C, D are points around a circle in that order. Then $\angle ABD = \angle ACD$. Similarly, $\angle BCA = \angle BDA$. Likewise, $\angle CDB = \angle CAB$. Finally, $\angle DAC = \angle DBC$. Alternatively, if two angles subtend the same arc, they are equal.
- **Opposite angles**: Again, suppose A, B, C, D are points around the circle. Then $\angle ABC + \angle ADC = 180°$, and $\angle BCD + \angle BAD = 180°$.

This review is important because finding or constructing four concyclic points is crucial when trying to solve a problem using Power of a Point.

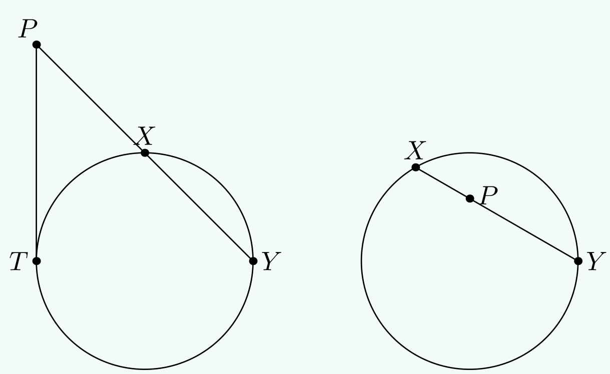
1 What is Power of a Point?

1.1 First Definition of Power of a Point

Theorem (First Definition of Power of a Point)

Suppose we have a circle ω and a point P. Draw a line through P intersecting the circle again, and suppose it intersects the circle at two points X and Y. Then we define the power of point P with respect to ω, denoted as $\text{Pow}(P, \omega)$, as the product of the directed segments PX and PY.

- The first question you may ask is, what is a directed segment? It's a line segment with both length and direction. When you multiply two directed segments with the same direction, the sign of the product is positive; when you multiply two directed segments with opposing directions, the sign of the product is negative.
  - Can you determine when $\text{Pow}(P, \omega)$ is less than, equal to, and greater than zero?
- If the line drawn through P is a tangent to the circle at T, then we say that both intersection points are T. That is, in this case, the power of P is $PT \times PT = PT^2$.



1.2 Why is the power of a (fixed) point constant?

The most important and fundamental thing about the power of a point is that no matter what line you draw, as long as it intersects the circle, the product of the two directed segments it creates is constant. Let's prove this.

Proof (The Power is Constant)

There are a few cases. You've probably seen them all, but here they are:

Case I: P is outside ω.

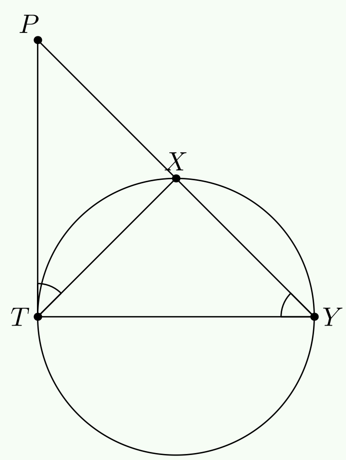
Let $\overline{PT}$ be a tangent from P to ω and $\overline{PXY}$ be an arbitrary secant. Because $\overline{PT}$ is a tangent, we have that $\angle PYT = \angle XYT = \angle XTP$. Furthermore, since $\angle TPX = \angle TPY$ (reflexive property), we must have by AA similarity that $\triangle PTX \sim \triangle PYT$. Therefore, we get the ratio $\frac{PT}{PX} = \frac{PY}{PT}$, which gives the $PX \times PY = PT^2$. Since PT is fixed given a fixed P, we must have that no matter which way you draw the secants through P, the power remains the same. (Notice in this case since the directed lengths PX and PY are in the same direction, the power of P is positive.)

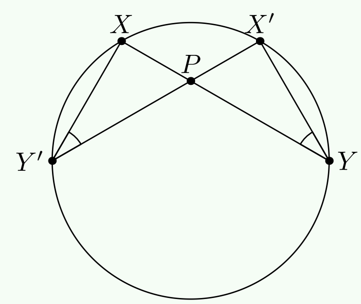
Case II: P is inside ω.

Consider any two arbitrary chords passing through P, $\overline{XPY}$ and $\overline{X'PY'}$ such that WLOG X' lies on the minor arc $\widehat{XY}$. Then we have that by definition, quadrilateral $XX'YY'$ is cyclic, so we must have that $\angle XY'P = \angle XY'X' = \angle XYX' = \angle PYX'$ and similarly $\angle Y'XP = \angle PX'Y$. Therefore, by AA similarity, we have that $\triangle XPY' \sim \triangle X'PY$. This gives $\frac{PX}{PX'} = \frac{PY'}{PY}$, so $PX \times PY = PX' \times PY'$ as desired. (Notice in this case since the directed lengths PX and PY are in different directions, the power of P is negative.)

Case III: P is on ω.

This case is trivial. Any line passing through P intersects ω at P as well, so the power must always be 0.





1.3 Power of a Point, Rewritten

§1.3.1 Statement

Theorem (Second Definition of the Power of a Point)

Suppose we have a circle ω with center O and a point P. We define the power of point P with respect to ω, $\text{Pow}(P, \omega) = d^2 - r^2$, where $d = OP$ and r the radius of ω. We claim that the two definitions of the power of a point are equivalent.

Proof (Equivalence of Definition of the Power of a Point)

Case I: P is outside ω.

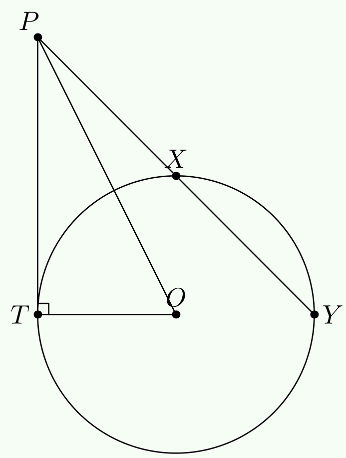
Let $\overline{PT}$ be a tangent from P to ω. Then we have that $d^2 - r^2 = OP^2 - OT^2 = PT^2$, as desired, since we proved using the first definition that $Pow(P, \omega) = PT^2 = PX \times PY$ for arbitrary secant $\overline{PXY}$ (notice that signs match; both give positive power).

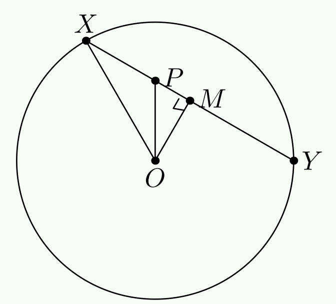
Case II: P is inside ω.

Consider $\overline{XY}$ be an arbitrary chord passing through P, with midpoint M. Then we have that $\overline{OM} \perp \overline{XMY}$ Therefore, we have that $d^2 = OP^2 = OM^2 + MP^2$ and $r^2 = OX^2 = OM^2 + XM^2$. Subtracting gives $d^2 - r^2 = MP^2 - XM^2 = (MP - XM) \times (MP + XM) = PX \times PY$, as desired (notice that signs match; both give negative power).

Case III: P is on ω.

This case is trivial. $d^2 - r^2 = OP^2 - OP^2 = 0$ as desired.





2 Exercises

Problem 1 (2020 AMC 12B)**.** In unit square $ABCD$, the inscribed circle ω intersects $\overline{CD}$ at M, and $\overline{AM}$ intersects ω at a point P different from M. What is AP?

Problem 2. Let ω and γ be two circles intersecting at P and Q. Let their common external tangent touch ω at A and γ at B. Prove that $\overline{PQ}$ passes through the midpoint M of $\overline{AB}$.

Problem 3 (2013 AMC 10A)**.** In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC?

Problem 4 (2019 AIME I)**.** In convex quadrilateral $KLMN$ side $\overline{MN}$ is perpendicular to diagonal $\overline{KM}$, side $\overline{KL}$ is perpendicular to diagonal $\overline{LN}$, $MN = 65$, and $KL = 28$. The line through L perpendicular to side $\overline{KN}$ intersects diagonal $\overline{KM}$ at O with $KO = 8$. Find MO.

Problem 5. In $\triangle ABC$, Let the perpendicular from B to AC intersect circle with diameter AC at points P and Q, and Let the perpendicular from C to AB intersect circle with diameter AB at points R and S. Prove that P, Q, R, S are concyclic.

Problem 6. Let C be a point on a semicircle of diameter $\overline{AB}$ and let D be the midpoint of arc $\widehat{AC}$. Let E be the projection of D onto the line BC and F the intersection of line AE with the semicircle. Prove that BF bisects the line segment $\overline{DE}$.