Trigonometry

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Trigonometry is a branch of geometry studying triangles using their angle measures.

1 Definitions

Before we delve into some of the theory, we have to define some terms first.

1.1 Trigonometry using Right Triangles

Definition 1 (Sine, Cosine, and Tangent)
Suppose $\triangle ABC$ is a right triangle with $\angle ACB = 90°$. Then we define $\sin \angle ABC$ as $\frac{AC}{AB}$, $\cos \angle ABC$ as $\frac{BC}{AB}$, and finally $\tan \angle ABC = \frac{AC}{BC}$.

Notice that $\tan \angle ABC = \frac{\sin \angle ABC}{\cos \angle ABC}$. Other trig functions are also defined, but they are not seen as often on competitions like the AMCs.

Definition 2 (Secant, Cosecant, and Cotangent)
In right $\triangle ABC$ with $\angle ACB = 90°$, we define $\sec \angle ABC$ as $\frac{AB}{BC}$, $\csc \angle ABC$ as $\frac{AB}{AC}$, and finally $\cot \angle ABC = \frac{BC}{AC}$.

Notice that these functions are just the reciprocals of sine, cosine, and tangent! In fact, if you know the sine and the cosine, the values of the rest of these trigonometric functions are known. To gain some more information about trig functions, we can use geometry:

Example 3
What is the value of $\sin^2 \theta + \cos^2 \theta$ if $\theta < 90°$?

We can use the Pythagorean Theorem to find this value.

Solution
Suppose $\theta = \angle ABC$ in right $\triangle ABC$ with $\angle ACB = 90°$. By definition, $\sin \angle ABC = \frac{AC}{AB}$ and $\cos \angle ABC = \frac{BC}{AB}$, so $\sin^2 \theta + \cos^2 \theta = \frac{AC^2+BC^2}{AB^2}$. Since $\triangle ABC$ is a right triangle, $AC^2 + BC^2 = AB^2$ by the Pythagorean Theorem! So, $\sin^2 \theta + \cos^2 \theta = 1$ whenever θ is an acute angle.

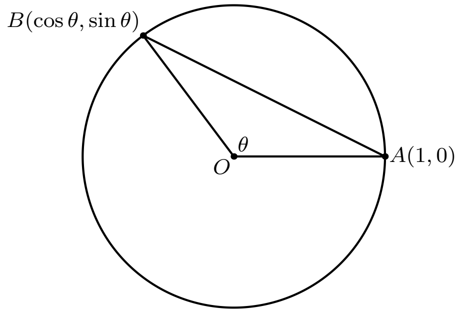
We can use this fact if we want to convert from $\sin \theta$ to $\cos \theta$ or vice versa.

Exercise 4. If $\sin \theta = \frac{\sqrt{2}}{2}$, then what is $\cos \theta$ if $\theta < 90°$?

We can extend the definitions of these trigonometric functions by using the unit circle.

1.2 Trigonometry using the Unit Circle

Let's generalize to include all angles, rather than just the angles from $0°$ to $90°$. Let's start with the relation $x^2 + y^2 = 1$ – this equation is true when $x = \cos\theta$ and $y = \sin\theta$. To extend the trigonometry definitions to all angles, let's *define* $\cos\theta$ *and* $\sin\theta$ *by this equation.*



In other words, we take the angle measure of arc AB, and we define $(\cos\theta, \sin\theta)$ as point B.

Exercise 5. Check that this new definition is compatible with the previous definition involving right triangles. In other words, check that $\cos\theta$ and $\sin\theta$, in the first quadrant, did not change based on the definition we used.

2 Trigonometric Identities

Besides the identity $\sin^2\theta + \cos^2\theta = 1$, there are other convenient identities for trig functions.

Exercise 6. Express the following in terms of $\sin(x)$ and $\cos(x)$:

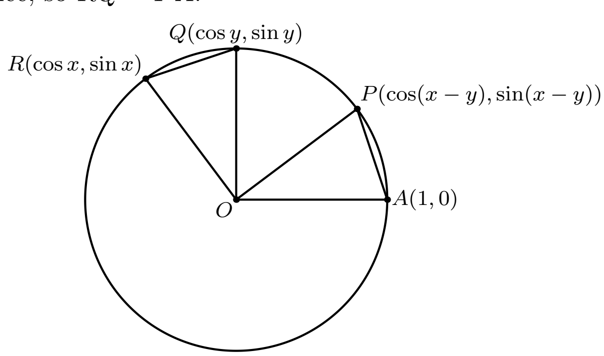
1. $\sin(-x)$ and $\cos(-x)$
2. $\sin(90 - x)$ and $\cos(90 - x)$
3. $\sin(180 - x)$ and $\cos(180 - x)$
4. $\sin(x + 90)$ and $\cos(x + 90)$

2.1 Angle Addition and Subtraction

The unit circle definition gives us a nice formula with angle subtraction:

Theorem 1 (Cosine Angle Subtraction Formula)
$\cos(x - y) = \cos x \cos y + \sin x \sin y$.

To prove that this formula is true, we can use the unit circle. The two triangles ORQ and OPA must be congruent by SAS congruence, so $RQ = PA$.



Applying the distance formula to RQ, we get

$$RQ = \sqrt{(\cos x - \cos y)^2 + (\sin x - \sin y)^2},$$

and we can expand this to get

$$RQ = \sqrt{2 - 2\cos x \cos y - 2\sin x \sin y}$$

since $\cos^2\theta + \sin^2\theta = 1$. On the other hand, applying the distance formula to PA gives

$$PA = \sqrt{(\cos(x-y) - 1)^2 + \sin^2(x-y)} = \sqrt{2 - 2\cos(x-y)}.$$

Setting these distance equal, we get that

$$\cos(x-y) = \cos x \cos y + \sin x \sin y.$$

Exercise 7. Use the cosine angle subtraction formula to find formulas for $\cos(x+y), \sin(x+y)$, and $\sin(x-y)$.

Exercise 8. Can you find a formula for $\tan(x+y)$?

2.2 Double Angle and Half Angle Formulas

Using the sine, cosine, and tangent addition formulas, we derive the **double angle** formulas:

$$\begin{aligned}\sin 2x &= 2\sin x \cos x\\ \cos 2x &= \cos^2 x - \sin^2 x\\ \tan 2x &= \frac{2\tan x}{1 - \tan^2 x}.\end{aligned}$$

The cosine double angle formula can be expressed in multiple ways:

$$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x,$$

and these equations allow us to derive **half-angle** formulas for sine and cosine. Solving for $\sin x$ and $\cos x$ in these equations gives us

$$\begin{aligned}\cos\frac{x}{2} &= \pm\sqrt{\frac{1+\cos x}{2}}\\ \sin\frac{x}{2} &= \pm\sqrt{\frac{1-\cos x}{2}}.\end{aligned}$$

The sign we take depends on the quadrant of $\frac{x}{2}$.

Exercise 9. What is the value of $\sin(22.5)$?

2.3 Example Problem

This is an example of a trigonometry question that appeared on a past AMC 12.

Problem 1 (AMC 12)
If

$$\sum_{n=0}^{\infty} \cos^{2n}\theta = 5,$$

what is the value of $\cos 2\theta$?

Solution

The left hand side of the given equation is a geometric series. Using the infinite geometric series formula, we get

$$\frac{1}{1-\cos^2\theta} = 5,$$

and we get that $\cos^2\theta = \frac{4}{5}$. Using the cosine double angle formula, we get that

$$\cos 2\theta = 2\cos^2\theta - 1 = \boxed{\frac{3}{5}}.$$

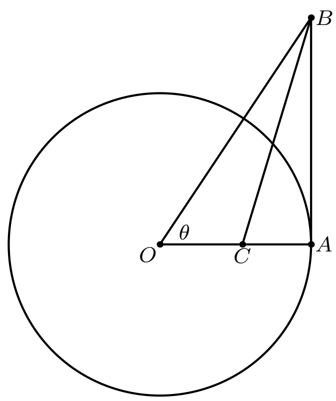
3 Some Problems

Problem 1. What is

$$\frac{\sin(1)\sin(2)\cdots\sin(89)}{\cos(1)\cos(2)\cdots\cos(89)}?$$

(All values are in degrees)

Problem 2 (AMC 12)**.** A circle centered at O has radius 1 and contains the point A. The segment AB is tangent to the circle at A and $\angle AOB = \theta$. If point C lies on $\overline{OA}$ and $\overline{BC}$ bisects $\angle ABO$, then what is OC? Your answer should be expressed in terms of trig functions.



Problem 3 (AMC 12)**.** Which of the following describes the largest subset of values of y within the closed interval $[0, \pi]$ for which

$$\sin(x+y) \le \sin(x) + \sin(y)$$

for every x between 0 and π, inclusive?

(A) $y = 0$ **(B)** $0 \le y \le \frac{\pi}{4}$ **(C)** $0 \le y \le \frac{\pi}{2}$ **(D)** $0 \le y \le \frac{3\pi}{4}$ **(E)** $0 \le y \le \pi$

Problem 4 (AMC 12)**.** The functions $\sin(x)$ and $\cos(x)$ are periodic with least period 2π. What is the least period of the function $\cos(\sin(x))$?

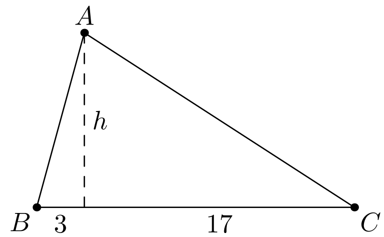
Problem 5 (AMC 12)**.** A quadrilateral is inscribed in a circle of radius $200\sqrt{2}$. Three of the sides of this quadrilateral have length 200. What is the length of the fourth side?

Problem 6 (AMC 12)**.** Alice and Bob live 10 miles apart. One day Alice looks due north from her house and sees an airplane. At the same time Bob looks due west from his house and sees the same airplane. The angle of elevation of the airplane is 30° from Alice's position and 60° from Bob's position. Which of the following is closest to the airplane's altitude, in miles?

(A) 3.5 **(B)** 4 **(C)** 4.5 **(D)** 5 **(E)** 5.5

Problem 7 (AMC 12)**.** A bee starts flying from point P_0. She flies 1 inch due east to point P_1. For $j \geq 1$, once the bee reaches point P_j, she turns 30° counterclockwise and then flies $j+1$ inches straight to point P_{j+1}. When the bee reaches P_{2015} she is exactly $a\sqrt{b} + c\sqrt{d}$ inches away from P_0, where a, b, c and d are positive integers and b and d are not divisible by the square of any prime. What is $a+b+c+d$?

Problem 8 (AIME)**.** In triangle ABC, $\tan \angle CAB = 22/7$, and the altitude from A divides BC into segments of length 3 and 17. What is the area of triangle ABC?



Problem 9 (AIME)**.** Given that $A_k = \frac{k(k-1)}{2} \cos \frac{k(k-1)\pi}{2}$, find $|A_{19} + A_{20} + \cdots + A_{98}|$.

Problem 10 (AIME)**.** Given that $(1+\sin t)(1+\cos t) = 5/4$ and

$$(1-\sin t)(1-\cos t) = \frac{m}{n} - \sqrt{k},$$

where k, m, and n are positive integers with m and n relatively prime, find $k+m+n$.