Angle Chasing

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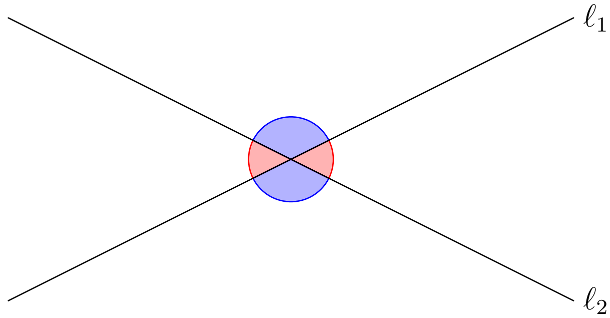
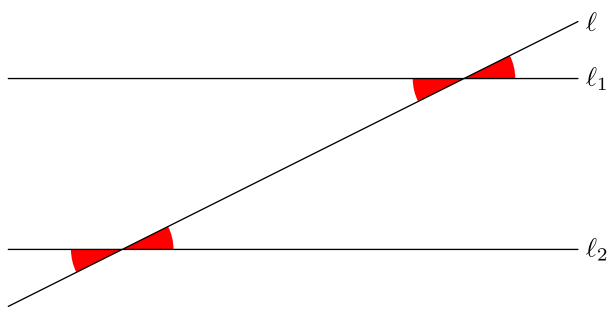
Angle chasing is the foundation of geometry, so we go in depth on the technique. Although it is built on very few rules, it can get very tricky at times.

1 Basic Facts



Fact 1

If ℓ_1 and ℓ_2 intersect at a point, then we split the 360° around this point into four angles, and opposite pairs of angles are equal to each other.



Fact 2

If $\ell_1 \parallel \ell_2$ and ℓ is a line not parallel to these two lines, then the acute angle formed by ℓ_1 and ℓ is congruent to the acute angle formed by ℓ_2 and ℓ.

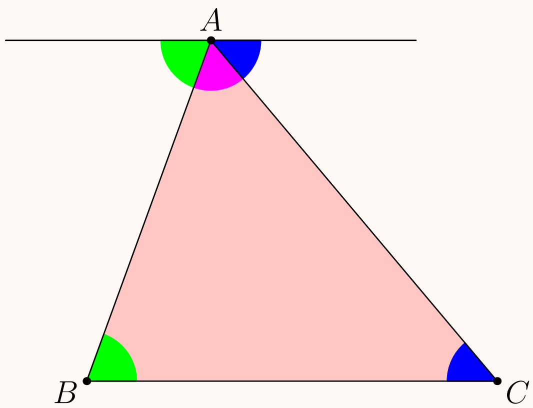
It is important to note that the converse of Fact 2 is true; that is, if ℓ intersects lines ℓ_1 and ℓ_2 and creates equal angles, we must have $\ell_1 \parallel \ell_2$.
Fact 2 is actually enough to prove one of the most important facts when it comes to angle chasing.

Example

Prove that the sum of the angles of any triangle ABC is 180°.

Proof

Draw a triangle and draw the line ℓ parallel to BC through A.



By Fact 2, we see that $\angle ACB$ is equal to the angle between AC and ℓ (i.e. the blue angles are equal). Similarly, we see that $\angle CBA$ is equal to the angle between AB and ℓ (i.e. the green angles are equal). Then the sum of the angles of the triangle is equal to the angle of a line, which is just 180°.

In fact, a much more general result is true.

Theorem (Sum of angles in a polygon with n sides)

The sum of the angles in an n-gon (a polygon with n sides) is $180^\circ(n-2)$.

The proof of this fact is a lot harder than the above proof. If you want, try to prove this fact! (Hint: Show that you can partition any polygon into triangles.)

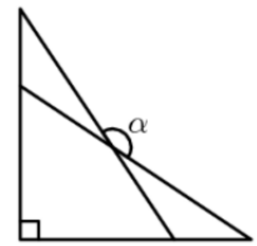
2 Some Exercises

Problems

Exercise 1 (2019 DMI Marathon/1)**.** In a quadrilateral, the angles form a geometric sequence with common ratio 2019. Compute the average of all the angles in the quadrilateral.

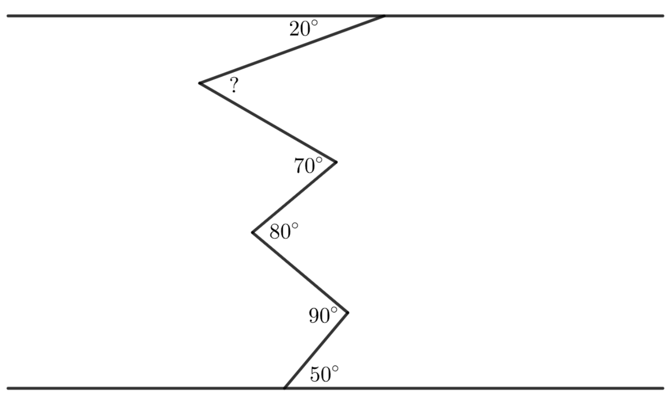
Exercise 2 (2020 AMC 10B/4)**.** The acute angles of a right triangle are a° and b°, where $a > b$ and both a and b are prime numbers. What is the least possible value of b?

Exercise 3 (2019 CMIMC Geometry/1)**.** The figure below depicts two congruent triangles with angle measures $40^\circ, 50^\circ$, and 90°. What is the measure of the obtuse angle α formed by the hypotenuses of these two triangles?



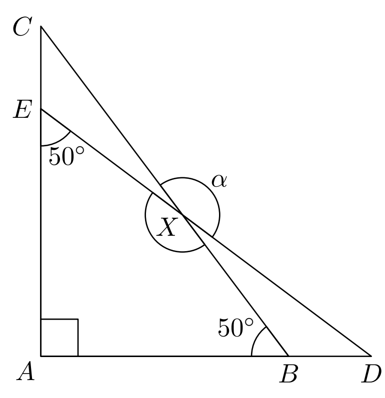
Exercise 4 (2018 CMIMC Geometry/1). Let ABC be a triangle. Point P lies in the interior of $\triangle ABC$ such that $\angle ABP = 20^\circ$ and $\angle ACP = 15^\circ$. Compute $\angle BPC - \angle BAC$.

Exercise 5. In the following diagram, what is the angle labeled with "?"? (The two long lines are parallel.)



Solutions

1. The sum of the angles of any quadrilateral is $180^\circ \cdot 2 = 360^\circ$, so the average of the angles is $\frac{360^\circ}{4} = \boxed{90^\circ}$.
2. Because the triangle is a right triangle, one of its angles is 90°. Since the sum of the angles of any triangle is 180°, we get
$$90^\circ + a^\circ + b^\circ = 180^\circ \implies a + b = 90.$$
Thus, a and b are two prime numbers that sum up to 90. We try $b = 2, 3, 5$, but these give $a = 88, 87, 85$, none of which are prime. If $b = 7$ then $a = 83$, which is prime. Thus, the least possible value of b is $\boxed{7}$.
3. First we label the points.

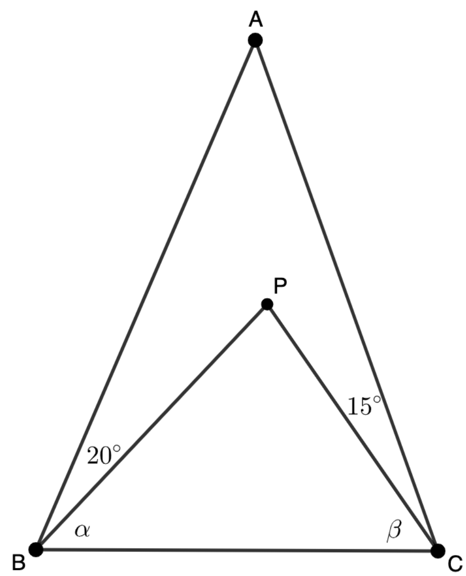


Note that because $AC > AE = AB$, we must have $\angle ABC > \angle ACB$, so $\angle ABC = 50°$. Similarly, $\angle AED = 50°$. Now we look at quadrilateral $ABXE$. The sum of the angles in this quadrilateral must be 360°, so

$$90° + 50° + 50° + \angle BXE = 360° \implies 190° + \angle BXE = 360° \implies \angle BXE = 170°.$$

By vertical angles, we have $\angle BXE = \angle CXD = \alpha$, so $\alpha = \boxed{170°}$.

4. First we draw a diagram.



We label $\angle PBC = \alpha$ and $\angle PCB = \beta$. Looking at $\triangle BPC$, we see

$$\alpha + \beta + \angle BPC = 180° \implies \angle BPC = 180° - \alpha - \beta.$$

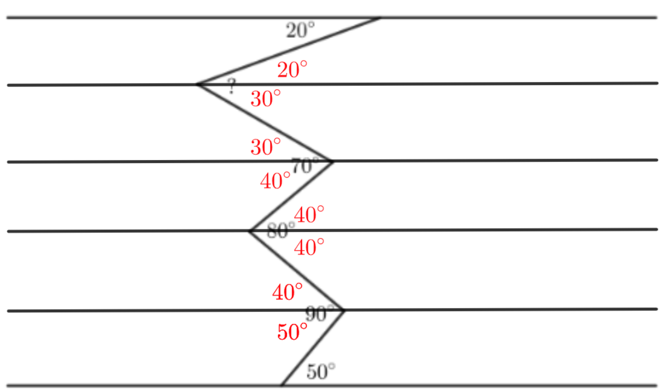
Looking at $\triangle ABC$, we see

$$(\alpha + 20°) + (\beta + 15°) + \angle BAC = 180° \implies \angle BAC = 145° - \alpha - \beta.$$

Subtraction our two equations yields

$$\angle BPC - \angle BAC = (180° - \alpha - \beta) - (145° - \alpha - \beta) = \boxed{35°}.$$

5. We draw parallel lines through all of the vertices.



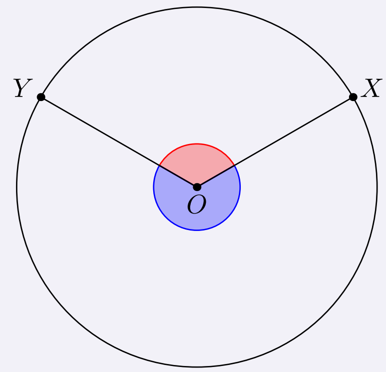
We use Fact 2 several times to get all of the red angles. Then the answer is $20° + 30° = \boxed{50°}$.

3 Circles and Angles

First we define what we mean by the measure of arc $\widehat{XY}$.

Definition

For points X and Y on a circle with center O, then the measure of arc $\widehat{XY}$ (often shortened as $\widehat{XY}$) is defined to be either $\angle XOY$ or the reflex angle $\angle XOY$, depending on context.



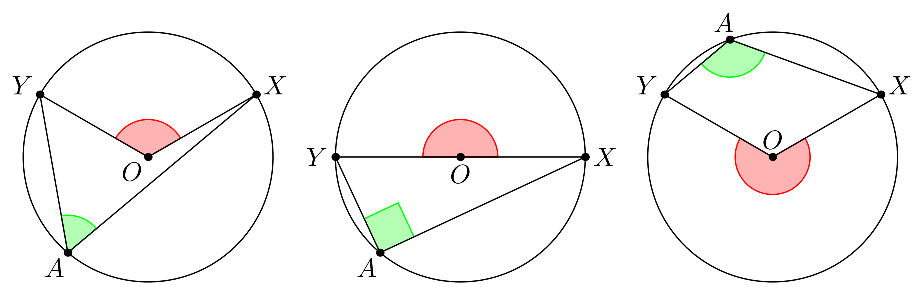
The smaller of the two arcs is called "minor arc $\widehat{XY}$", while the other one is called "major arc $\widehat{XY}$".

With this definition, we can state the inscribed angle theorem.

Theorem (Inscribed Angle Theorem)

If X, A, and Y are points on a circle centered at O, then $\angle XAY$ is equal to half of $\widehat{XY}$, where we choose either the normal angle or the reflex angle so that it and $\angle XAY$ "point in the same direction".

The statement of the theorem isn't entirely clear, so here are a few examples.



To prove the inscribed angle theorem, we require one more fact.

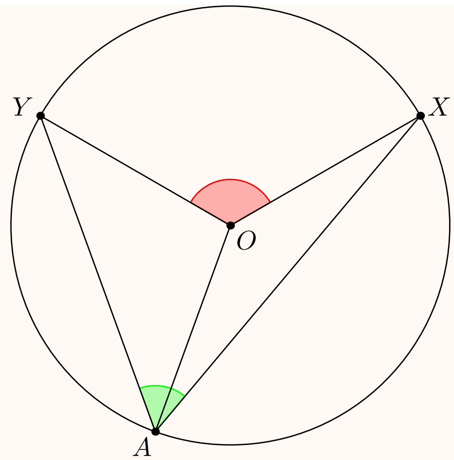
Fact 3

If $\triangle ABC$ is isosceles with $AB = AC$, then $\angle ABC = \angle ACB$.

Now we can prove the inscribed angle theorem.

Proof

There are 3 cases: when A is on minor arc XY, major arc XY, or if XY is a diameter. All three cases are similar, so we just show the first case. You should complete the other two cases though.



Label $\angle OAY = \alpha$ and $\angle OAX = \beta$. Since $OA = OY$, we have $\angle OAY = \angle OYA = \alpha$. Since the sum of the angles of $\triangle OAY$ is $180°$, we must have $\angle AOY = 180° - 2\alpha$. Similarly, $\angle AOX = 180° - 2\beta$. Then

$$\angle XOY = 360° - (\angle AOY + \angle AOX) = 360° - ((180° - 2\alpha) + (180° - 2\beta)) = 2(\alpha + \beta) = 2\angle XAY$$

as desired.

Note that this implies that if A lies on a circle with diameter XY, then $\angle XAY = 90°$.

4 More Exercises

Exercise 1. If A, B, C, D lie on a circle in that order, prove that $\angle BAC = \angle BDC$. ($ABCD$ is called a **cyclic quadrilateral**)

Exercise 2. If A, B, C, D lie on a circle in that order, prove that $\angle ABC + \angle CDA = \angle DAB + \angle BCD = 180°$.

Exercise 3. Suppose A, B, C, D lie on a circle such that AC and BD intersect inside the circle at a point P. Show that $\angle APB = \frac{\widehat{AB} + \widehat{CD}}{2}$.

Exercise 4. Suppose A, B, C, D lie on a circle such that the extension of AB past B and the extension of CD past C intersect outside the circle at a point P. Show that $\angle BPC = \frac{\widehat{AD} - \widehat{BC}}{2}$

Exercise 5 (Reim's Theorem)**.** Let ω_1 and ω_2 be two circles that intersect at X and Y. Draw a line through X that intersects ω_1 at A and ω_2 at B. Draw a line through Y that intersects ω_1 at C and ω_2 at D. Prove that $AC \parallel BD$.

Exercise 6 (HARD)**.** Prove that the converse of the statements in exercises 1 and 2 hold.

5 Triangle Centers

Before we continue, we introduce some notation for triangle $\triangle ABC$. We say $\angle BAC = A, \angle ACB = C, \angle CBA = B$. This shorthand allows us to express many angles more concisely.
Now we define three important triangle centers. We take for granted that they exist.

Definition 7 (Triangle Centers)

We define the **circumcenter** of $\triangle ABC$ to be the point that is the center of the circle that passes through the points A, B, and C. This is point is commonly denoted by O, while the circle through A, B, and C is called the **circumcircle** of $\triangle ABC$.

We define the **incenter** of $\triangle ABC$ to be the point that is the intersection of the angle bisectors of $\angle BAC, \angle ACB$, and $\angle CBA$. This is commonly denoted by I. There exists a circle centered at I that is tangent to all three sides of the triangle. This circle is called the **incircle** of $\triangle ABC$.
We define the **orthocenter** of $\triangle ABC$ to be the point that is the intersection of the altitudes from A, B, and C to BC, CA, and AB respectively. This is commonly denoted H.

Given these definitions, do the following.

Example

Let $\triangle ABC$ be an acute triangle. Compute in terms of A, B, and C:

- $\angle BOC$
- $\angle BIC$
- $\angle BHC$

Example

Do the above except when $A \geq 90°$.

6 Problems

Remember to look at exercise 6 in section 4 in order to prove a quadrilateral is cyclic!

Problem 1. If $\angle BHC = \angle BIC = \angle BOC$ where H, I, and O are the orthocenter, incenter, and circumcenter of $\triangle ABC$, respectively, find A.

Problem 2. Let $\triangle ABC$ be an acute triangle. Let D, E, and F be the feet from A, B, and C to BC, CA, and AB respectively. Let H be the orthocenter of $\triangle ABC$. Find as many cyclic quadrilaterals as you can! (Hint: There are 6.)

Problem 3. Using the above notation, prove that H is the incenter of $\triangle DEF$. (Hint: Use some of the cyclic quadrilaterals from Problem 2. Also look at exercise 1 in section 4.)

Problem 4 (2019 AMC10A/13)**.** Let $\triangle ABC$ be an isosceles triangle with $BC = AC$ and $\angle ACB = 40°$. Contruct the circle with diameter $\overline{BC}$, and let D and E be the other intersection points of the circle with the sides $\overline{AC}$ and $\overline{AB}$, respectively. Let F be the intersection of the diagonals of the quadrilateral $BCDE$. What is the degree measure of $\angle BFC$?

Problem 5. Let H be the orthocenter of acute $\triangle ABC$. Let H_A be the reflection of H over BC. Prove that H_A lies on the circumcircle of $\triangle ABC$.

Problem 6. Let H be the orthocenter of acute $\triangle ABC$. Let H'_A be the reflection of H over the midpoint of BC. Prove that H'_A lies on the circumcircle of $\triangle ABC$.