Combinatorics

30 September 2016

1. Find the number of 10-unit paths from (0,0) to (6,4) along the lattice.
2. Find the number of 10-letter strings that consist of 7 x's and 3 y's.
3. Find the number of ordered triples (x, y, z) where x, y, and z are distinct integers between 1 and 10 inclusive.
4. Find the number of sets $\{x, y, z\}$ where x, y, and z are distinct integers between 1 and 10 inclusive.
5. Find the number of ordered triples (x, y, z) where x, y, and z are non-negative integers that satisfy $x + y + z = 12$.
6. Find the number of ordered triples (x, y, z) where x, y, and z are positive integers that satisfy $x + y + z = 12$.
7. Find the number of ordered triples (x, y, z) where x, y, and z are positive integers that satisfy $x + y + 2z = 20$.
8. Find the number of ordered triples (x, y, z) where x, y, and z are integers that satisfy $x + y + z = 8$ and $x \geq 1$, $y \geq 2$, and $z \geq 3$.
9. Find the coefficient of x^7y^3 in the expansion of $(x + y)^{10}$.
10. Find the sum of the coefficients in the expansion of:
    (a) $(x + y)^{10}$
    (b) $(x - y)^{100}$
    (c) $(2x + 3y - 3z)^7$
11. Find the 12^{th} term of $(2x - 1)^{13}$.
12. Find the 7^{th} term of $(\frac{4x}{5} - \frac{1}{x^3})^9$.
13. Find the constant term in the expansion of $(\frac{3}{2}x^2 - \frac{1}{3x})^9$.
14. Find the coefficient of x^5 in the expansion of $(x^2 + 2x + 3)^5$.
15. Find the number of terms in the expansion of $(x + y)^{12}$.
16. Find the number of terms in the expansion of $(x + y + z)^{12}$.
17. Find the number of 5-letter strings in which the letter A appears exactly once.

18. Find the number of 5-letter strings in which the letter A appears at least once.

19. Find the number of integers greater than 53000 that have distinct digits.

20. Find the number of integers greater than 53000 that have distinct digits and do not contain a 0 or 9.

21. Twenty points lie in a plane with no two points collinear. Find the number of line segments that have two of these points as endpoints.

22. Twenty points lie in a plane with no two points collinear. Find the number of triangles that three of these points as vertices.

23. How many 4-digit numbers are there consisting of the digits 1,2, and 3, in which each digit is less than or equal to the digit on its right.

24. How many integers from 1 to 1000 are neither perfect squares, perfect cubes, nor perfect fourth powers.

25. How many 9-digit positive integers have the property that the digits do not decrease from left to right.

26. Ten points are marked on the circumference of a circle. How many distinct convex polygons of three or more sides can be drawn using some or all of the points as vertices?

27. Ten different paintings are to be allocated to n office rooms so that no room gets more than one painting. Find the number of ways of accomplishing this if (a) $n = 14$ and (b) $n = 6$.

28. Solve the problem above if there are ten identical posters instead of ten distinct paintings.

29. If there are 10 people at a meeting and everyone shakes hands with everyone else, how many handshakes will ther be?

30. If there are n couples at a party and everyone shakes hands with everyone else except his or her partner, how many handshakes will there be?

31. Given ten integers, prove that there are two integers whose difference is divisible by nine.

32. A certain fruit stand has 25 apples of 3 different varieties . Prove that at least 9 of the apples must be of the same variety.

33. Given 8 positive integers each less than 15, prove that among there pairwise non-negative differences there are 3 equal differences.

34. Prove that from a set of 10 distinct 2-digit numbers it is possible to select two disjoint subsets whose members have the same sum.

35. Consider a tournament in which each of n teams plays every other team and each team wins at least once. Show that there are at least two teams having the same number of wins.

36. If 5 points are chosen at random in the interior of an equilateral triangle each side of which is 2 units long, show that at least one pair of points has a separation of less than 1 unit.

37. Daphne is visited periodically by her three best friends: Alice, Beatrix, and Claire. Alice visits every third day. Beatrix visits every fourth day. Alice visits every fifth day. Today all three friends visited Daphne. How many days of the next 365-day period will exactly two of the friends visit Daphne?

38. Find the number of sets of two or more consecutive positive integers whose sum is 100.

39. If $i^2 = -1$, compute $(1 + i)^{20} - (1 - i)^{20}$.

40. How many distinct positive integral divisors has $(30)^4$.

41. A fair die is rolled six times. Find the probability of rolling at least a 5 at least 5 times.

42. How many ways are there to roll a total of twelve using three fair dice.

43. Three fair dice are tossed. What is the probability that the three numbers facing up can be arranged in arithmetic progression with a common difference of 1.

44. A six digit number (base 10) is *squarish* if it satisfies the following conditions:

    (a) none of its digits is zero;
    (b) it is a perfect square; and
    (c) the first two digits, the middle two digits, and the last two digits are all perfect squares when considered as two digit numbers.

    How many *squarish* numbers are there?

45. How many integers with four different digits between 1000 and 9999 are there such that the absolute value of the difference between the first and last digit is 2?

46. Computer the number of perfect cubes from 1 to 500,000 inclusive that are multiples of 7.

47. Compute the number of integers between 100,000 to one million with the property that their digits are distinct and increasing from left to right.

48. You have 121 marbles, some of which are red, some of which are white, and the rest blue. You also have 10 jars. If all of the marbles are distributed into the jars, then there must be a jar with at least n marbles of the same color. Compute the maximum possible value for n.

49. The number of different 10-letter words that can be made form the letters of the work REASSESSES is the same as the number of x-letter words than can be made from the letters of the word REDUCTIONS. Compute x. *Note: the term* word *refers to any arrangement of letters.*

50. Find the number of 10-unit paths from (0,0,0) to (3,3,4) along a 3-dimensional lattice.