New York City Math Team

Team Round - October 16, 2015

1.) If $\sum_{k=1}^{\infty} kx^k = 30, x \in \mathbb{R}$, compute x

2.) Given $x, y \in \mathbb{R}$ satisfy $x^4 + x^2y^2 + y^4 = 72$.
Compute the minimum possible value of $2x^2 + xy + 2y^2$

3.) In $\Delta ABC, AB = 137, AC = 241$ and $BC = 200$. Let D be a point on $\overline{BC}$ such that the incircles of ΔABD and ΔACD meet $\overline{AD}$ at the same point E. Compute CD

4.) Hilary and Bernie are betting on a roll of a standard pair of 6-sided dice. Hilary bets that a sum of 12 will occur first, while Bernie bets that two consecutive sums of 7 will appear first. They continue to roll the dice until one person wins. Compute the probability that Hilary wins.

5.) Compute the radius of the largest possible circle that lies above the x-axis and below the parabola $y = 2 - x^2$

6.) The sequence of numbers $t_1, t_2, t_3, \ldots$ is defined by $t_1 = 2$ and $t_{n+1} = \frac{t_n - 1}{t_n + 1}, \forall n \in \mathbb{N}$.
Compute t_{999}

7.) Given that 3^{11} can be expressed as the sum of k consecutive integers, compute the largest possible value of k.

8.) The three roots of $30x^3 - 50x^2 + 22x - 1$ are distinct real numbers between 0 and 1.
Compute $s_0 + s_1 + s_2 + \ldots$, where s_n denotes the sum of the nth powers of the three roots for all non-negative integers n.

9.) For each integer $n \geq 4$, let a_n denote the base-n number $\overline{.133}_n$.
You are given that the product $a_4 a_5 \ldots a_{99}$ can be rewritten as $\frac{m}{n!}$, where m and n are positive integers and n is as small as possible. Compute the value of m.

10.) Given complex numbers z and w such that $|w| = 3$ and $z = w - \frac{1}{w}$. The set of all such points z (call it T) form a closed curve in the complex plane. Compute the area inside T.

11.) Amy places 100 pennies on a table, 30 showing heads and 70 showing tails. She chooses 40 pennies at random (all different), and turns them over. That is, if a chosen penny was heads, it is now tails and if it was tails, it is now heads. At the end, what is the expected number of pennies showing heads?

12.) Find all non-zero real numbers x such that $\log_2(x+2), \log_4(3x+4)$ and $\log_8(7x+8)$ are in arithmetic progression (in this order).

13.) If a, b, c are positive integers with $a < b < c$, $a+b+c = 79$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{1}{5}$, compute c.

14.) Two circles C_1 and C_2 have a common chord $\overline{AB}$. Let P be a point on C_1 that is outside C_2. The line segments $\overline{PA}$ and PB are extended to meet C_2 at X and Y respectively. If $AB = 6, AP = 5, BP = 7$ and $AX = 16$, compute XY.

15.) Given a quadratic function P such that $P(0) = 7, P(1) = 10$ and $P(2) = 25$.
You are also given that a, b and c are integers such that every positive number $x < 1$ satisfies the following: $\displaystyle\sum_{n=0}^{\infty} P(n)x^n = \frac{ax^2+bx+c}{(1-x)^3}$
Compute the ordered triple (a, b, c)

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Answers to Team Round

1.) $\frac{5}{6}$

2.) $6\sqrt{6}$

3.) 152

4.) $\frac{7}{13}$

5.) $\frac{-1+2\sqrt{2}}{2}$

6.) $-\frac{1}{2}$

7.) 486

8.) 12

9.) 962

10.) $\frac{80\pi}{9}$

11.) 46

12.) $-1, \frac{-8+2\sqrt{2}}{7}$

13.) 42

14.) 18

15.) $(16, -11, 7)$