ARML 2006 (NYC Practice Session) Power Question 60 Minutes

The numbers in square brackets represent the point value for each item.

PALINDROMIC NUMBERS

Definitions

Palindromic numbers (***palindromes***) are natural numbers that read (in some base) the same forward and backward. Below we are interested only in palindromic numbers to base 10, and all numbers below are written in base-10 representation (leading zeroes are not allowed). The following numbers are palindromes: 5, 99, 121, 4774, and the following ones are not: –11, 1121, 5.5, 010. As usual, $\boldsymbol{N}$ is a set of all natural numbers, $|\boldsymbol{A}|$ is the number of elements in a set $\boldsymbol{A}$. We define $\boldsymbol{P}$ as a set of all palindromic numbers, and $\boldsymbol{P}_n$ as a set of all n-digit palindromic numbers, $n \in \boldsymbol{N}$.

Problems

1. a) Find (with proof) $|\boldsymbol{P}_n|$. [2]

   b) Find (with proof) the number of palindromes from $\boldsymbol{P}_n$ that are divisible by 3. [2]

2. a) Prove that every palindrome from $\boldsymbol{P}_{2k}$ ($k \in \boldsymbol{N}$) is divisible by 11. [2]

   b) Prove that more than 50% of palindromes from $\boldsymbol{P}_1 \cup \boldsymbol{P}_2 \cup \ldots \cup \boldsymbol{P}_{2k}$ are divisible by 11, $k \in \boldsymbol{N}$, $k \geq 2$. [2]

3. a) Find $a \notin \boldsymbol{P}$ such that $a^2 \in \boldsymbol{P}$ ($a \in \boldsymbol{N}$). [1]

   b) Find $a \notin \boldsymbol{P}$ such that $a^3 \in \boldsymbol{P}$ ($a \in \boldsymbol{N}$). [3]

4. a) Prove that there are infinitely many $p \in \boldsymbol{P}$ such that $p^2 \notin \boldsymbol{P}$. [2]

   b) Prove that there are infinitely many $p \in \boldsymbol{P}$ such that $p^2 \in \boldsymbol{P}$. [2]

5. a) Prove that there exists $p \in \boldsymbol{P}$ such that p is divisible by 2006. [2]

   b) Prove that there exists $p \in \boldsymbol{P}$ such that p is divisible by 2^{2006}. [2]

6. Find (with proof) all solutions of the equation $x + y = 2006$, $x, y \in \boldsymbol{P}$. [4]

7. Find (with proof) all solutions of the equation $x - y = 2006$, $x, y \in \boldsymbol{P}$. [4]

We assume that all sequences considered below contain only natural numbers. A Fibonacci-like sequence is a sequence $\{a_n\}$, $n \in \boldsymbol{N}$, that satisfies the Fibonacci condition $a_{n+2} = a_{n+1} + a_n$, $n \in \boldsymbol{N}$.

8. a) Prove that every infinitely increasing arithmetic sequence contains at least one element that is not a palindromic number. [2]

   b) Prove that every infinitely increasing Fibonacci-like sequence contains at least one element that is not a palindromic number. [2]

Some natural numbers could be represented as a sum of two palindromes (19 = 11 + 8), some others could not (21).

9. Prove that there are infinitely many natural numbers that could not be represented as a sum of two palindromes. [4]

For a given natural number n, we define $P(n)$ as the number of palindromes that are not greater than n:

$$P(n) = |\{p \in \boldsymbol{P}: p \leq n\}|.$$

10. Prove that $\lim_{n \to +\infty} \frac{P(n)}{n} = 0$. [4]

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Answers and Solutions

1. a) [2] $9\cdot 10^{[(n-1)/2]}$, where $[x]$ is the greatest integer less than or equal to x. For $n = 1$ all 1-digit natural numbers are palindromes, $p = \overline{a_1}$, $1 \le a_1 \le 9$ (9 choices), $|\boldsymbol{P}_1| = 9$. For $n = 2$ both digits should be the same, $p = \overline{a_1 a_1}$, $1 \le a_1 \le 9$ (9 choices), $|\boldsymbol{P}_2| = 9$ (the above cases could be part of generic cases below). For $n = 2k + 1$, $k \ge 1$: $p = \overline{a_1 \ldots a_k a_{k+1} a_k \ldots a_1}$, $1 \le a_1 \le 9$ (9 choices), $0 \le a_2 \le 9$ (10 choices), …, $0 \le a_{k+1} \le 9$ (10 choices), and all these digits are independent from each other, $|\boldsymbol{P}_{2k+1}| = 9\cdot 10^k$. For $n = 2k + 2$, $k \ge 1$: $p = \overline{a_1 \ldots a_k a_{k+1} a_{k+1} a_k \ldots a_1}$, $1 \le a_1 \le 9$ (9 choices), $0 \le a_2 \le 9$ (10 choices), …, $0 \le a_{k+1} \le 9$ (10 choices), and all these digits are independent from each other, $|\boldsymbol{P}_{2k+2}| = 9\cdot 10^k$. We can also note that each palindrome from $\boldsymbol{P}_{2k}$ ($\boldsymbol{P}_{2k-1}$) is uniquely identified by its number of digits and its "left-half" that could be any k-digit natural number, so $|\boldsymbol{P}_{2k}| = |\boldsymbol{P}_{2k-1}| = 9\cdot 10^{k-1}$.

   b) [2] $3\cdot 10^{[(n-1)/2]} = |\boldsymbol{P}_n|/3$. With the preceding notation, for $n = 1$ $a_1 = 3$, 6, or 9 (3 choices), for $n = 2$ $a_1 = 3$, 6, or 9 (3 choices). For $n = 2k + 1$, $k \ge 1$: $2a_1 + (2a_2 + \ldots + 2a_k + a_{k+1})$ is divisible by 3, so choices for $a_2, \ldots, a_{k+1}$ are independent (as above), and $a_1 = 3$, 6, or 9, or $a_1 = 2$, 5, or 8, or $a_1 = 1$, 4, or 7, depending on the remainder of $(2a_2 + \ldots + 2a_k + a_{k+1}) \bmod 3$. For $n = 2k + 2$, $k \ge 1$: $2a_1 + (2a_2 + \ldots + 2a_k + 2a_{k+1})$ is divisible by 3, so choices for $a_2, \ldots, a_{k+1}$ are independent (as above), and $a_1 = 3$, 6, or 9, or $a_1 = 2$, 5, or 8, or $a_1 = 1$, 4, or 7, depending on the remainder of $(2a_2 + \ldots + 2a_k + 2a_{k+1}) \bmod 3$. So, the only difference from solution of the Problem 1a) is that number of choices for a_1 is 3 instead of 9, for every other digit we still have 10 (independent) choices, and $|\boldsymbol{P}_{2k+1}| = |\boldsymbol{P}_{2k+2}| = 3\cdot 10^k$.

2. a) [2] Proof: we can simply apply the well-known criteria of divisibility by 11: "A natural number is divisible by 11 if and only if the alternating sum of its digits (from right to left) is divisible by 11" to a ($2k$)-digit palindrome, the above sum would be 0 (an even (odd) place from the right is an odd (even) place from the left, and digit on place l from the right is the same as digit on place l from the left, $1 \le l \le 2k$). Or we can prove the result directly: let's consider a ($2k$)-digit palindrome $p = \overline{a_1 \ldots a_k a_k \ldots a_1}$, $p = a_1 \cdot 10^0 + a_2 \cdot 10^1 + \ldots + a_k \cdot 10^{k-1} + a_k \cdot 10^k + \ldots a_2 \cdot 10^{2k-2} + a_1 \cdot 10^{2k-1}$, $p = a_1(10^{2k-1} + 10^0) + a_2(10^{2k-2} + 10^1) + \ldots a_k(10^k + 10^{k-1})$, $p = a_1(10^{2k-1} + 1^{2k-1}) + 10a_2(10^{2k-3} + 1^{2k-3}) + \ldots 10^{k-1}a_k(10^1 + 1^1)$. For every odd natural number l $10^l + 1^l$ is divisible by $10 + 1 = 11$, so every ($2k$)-digit palindrome p is divisible by 11.

   b) [2] Proof: from the result of the Problem 1a) we conclude that $|\boldsymbol{P}_1| = |\boldsymbol{P}_2|$, $|\boldsymbol{P}_3| = |\boldsymbol{P}_4|$, …, $|\boldsymbol{P}_{2k-1}| = |\boldsymbol{P}_{2k}|$. Let's consider sets $\boldsymbol{A} = \boldsymbol{P}_1 \cup \boldsymbol{P}_3 \cup \ldots \cup \boldsymbol{P}_{2k-1}$ and $\boldsymbol{B} = \boldsymbol{P}_2 \cup \boldsymbol{P}_4 \cup \ldots \cup \boldsymbol{P}_{2k}$, $|\boldsymbol{A}| = |\boldsymbol{B}|$. From the Problem 2a) we obtain that all palindromes from $\boldsymbol{B}$ are divisible by 11 (so at least 50% of palindromes from $\boldsymbol{A} \cup \boldsymbol{B}$ are divisible by 11), but some palindromes from $\boldsymbol{P}_3 \subset \boldsymbol{A}$ (121, 242, etc.) are also divisible by 11 (so strictly more than 50% of palindromes from $\boldsymbol{A} \cup \boldsymbol{B}$ are divisible by 11).

3. a) [1] 26 ($26 \notin \boldsymbol{P}$, $26^2 = 676 \in \boldsymbol{P}$). It could be found without calculator.

   b) [3] 2201 ($2201 \notin \boldsymbol{P}$, $2201^3 = 10662526601 \in \boldsymbol{P}$). It could be found with calculator.

4. a) [2] Proof: we will prove that for every $n \in \boldsymbol{N}$ the n-digit number $a_n = 10^n - 1$ is a palindrome, but its square is not a palindrome. Indeed, $10^n - 1 = \underbrace{\overline{9\ldots9}}_{n} \in P$, $(10^n - 1)^2 = 10^{2n} - 2\cdot 10^n + 1 = \overline{\underbrace{9\ldots9}_{n-1}8\underbrace{0\ldots0}_{n-1}1} \notin P$. Since all numbers a_n ($n \in \boldsymbol{N}$) are distinct, there are infinitely many of them.

   b) [2] Proof: we will prove that for every $n \in \boldsymbol{N}$ the ($n + 1$)-digit number $a_n = 10^n + 1$ is a palindrome, and its square is a palindrome too. Indeed, $10^n + 1 = \overline{1\underbrace{0\ldots0}_{n-1}1} \in P$, $(10^n + 1)^2 = 10^{2n} + 2\cdot 10^n + 1 = \overline{1\underbrace{0\ldots0}_{n-1}2\underbrace{0\ldots0}_{n-1}1} \in P$. Since all numbers a_n ($n \in \boldsymbol{N}$) are distinct, there are infinitely many of them.

5. a) [2] Proof: let's consider the following 2007 palindromes: $a_1 = 2, a_2 = 22, a_{2007} = \underbrace{\overline{2\ldots2}}_{2007}$. By the Pigeonhole Principle, at least two of them (a_n and a_m, $n > m$) have the same remainders mod 2006, so their difference is divisible by

2006: $a_n - a_m = a_{n-m} \cdot 10^m \vdots 2006 = 1003 \cdot 2$. Since 10 (and therefore 10^m) is relatively prime with 1003, a_{n-m} is divisible by 1003. But obviously a_{n-m} is divisible by 2 that is relatively prime with 1003. Therefore palindromic number a_{n-m} is divisible by $1003 \cdot 2 = 2006$.

b) [2] Proof: let's consider the natural number $M = 2^{2006}$ (its right-most digit is not equal to 0), and let k be the number of digits of M, $k < 2006$ ($2^2 = 4$ has less than 2 digits, and if we multiply any natural number by 2, the number of its digits would stay the same or increase just by 1). Let's also consider the following natural number N: its base-10 representation is "the same" as base-10 representation of M just written in the reversed order. Now let's consider the following palindrome $p = \overline{N\underbrace{0\ldots0}_{2006-k}M}$ and prove that p is divisible by 2^{2006}. Indeed, $p = N \cdot 10^{2006} + M$, 10^{2006} is divisible by 2^{2006}, $M = 2^{2006}$ is divisible by 2^{2006}, therefore p is also divisible by 2^{2006}.

6. [4] {(4; 2002), (2002; 4), (565; 1441), (1441; 565)}. Let's assume that there are $x, y \in \boldsymbol{P}$: $x + y = 2006$. WLOG we can assume that $x \geq y \geq 1$. Therefore $x < 2006$, $x \geq 2006/2 = 1003$, so $x \in \boldsymbol{P}_4$, and the x's left-most digit is either 1 or 2. If it is 2, x is a palindrome between 2000 and 2005, so $x = 2002$, $y = 2006 - x = 4$. If the x's left-most (and right-most) digit is 1, then the y's right-most (and left-most) digit is 5 ($x + y = 2006$). If $y = 5$, $x = 2001 \notin \boldsymbol{P}$, if $y = 55$, $x = 1951 \notin \boldsymbol{P}$, $y < 2006$, so y is a 3-digit palindrome, $y = \overline{5t5}$, $0 \leq t \leq 9$, $x = 2006 - y = 2006 - (505 + 10t) = 1501 - 10t \in \boldsymbol{P}$. $1411 \leq x \leq 1501$, $x \in \boldsymbol{P}$, so $x = 1441$ (the only palindromic number between 1411 and 1501), $t = 6$, $y = 565$. Obviously both pairs we found ((2002; 4) and (1441; 565)) are palindrome solutions of the equation $x + y = 2006$. Since this equation is symmetrical regarding x and y, condition $x \leq y$ leads to only two more solutions: (4; 2002) and (565; 1441).

7. [4] {(2662; 656)}. Let's assume that there are $x, y \in \boldsymbol{P}$: $x - y = 2006$, $x \in \boldsymbol{P}_n$, $y \in \boldsymbol{P}_m$, $n \geq m$, the x's left-most (and right-most) digit is $a \neq 0$, the y's left-most (and right-most) digit is $b \neq 0$. By comparing the right-most digits of x and $y + 2006 = x$ we obtain $a = b + 6$ or $a + 10 = b + 6$ ($b = a + 4$). Let's start from case $m \geq 5$. If $n = m$, by comparing the left-most digits of x and $y + 2006 = x$ we obtain $a = b$ or $a = b + 1$, that contradicts the above condition $a = b + 6$ or $b = a + 4$. If $n > m \geq 5$, then $a = 1$, $b = 9$ ($x = y + 2006$, 2006 is a 4-digit number), and this again contradicts the above condition $a = b + 6$ or $b = a + 4$. Now let's consider case $m = 4$. If $n = m$, by comparing the left-most digits of x and $y + 2006 = x$ we obtain $a = b + 2$ or $a = b + 3$, that contradicts the above condition $a = b + 6$ or $b = a + 4$. If $n > m = 4$, then again $a = 1$, $b = 5$ (from the above condition $a = b + 6$ or $b = a + 4$), therefore $y + 2006 \leq 5999 + 2006 < 10000 \leq x$, and this contradicts our assumption $x - y = 2006$. The last case is $m \leq 3$. In this case $x > 2006$, $x \leq 2006 + 999 = 3005$, $n = 4$, and $a = 2$ or $a = 3$. If $a = 3$ then $b = 7$, $x \leq 2006 + 797 < 3000$, that contradicts $a = 3$. If $a = 2$ then $b = 6$. If $y = 6$ ($m = 1$), $x = 2012 \notin \boldsymbol{P}$, if $y = 66$ ($m = 2$), $x = 2072 \notin \boldsymbol{P}$, so y is a 3-digit palindrome, $y = \overline{6t6}$, $0 \leq t \leq 9$, $x = 2006 + y = 2006 + (606 + 10t) = 2612 + 10t \in \boldsymbol{P}$. $2612 \leq x \leq 2702$, $x \in \boldsymbol{P}$, so $x = 2662$ (the only palindromic number between 2612 and 2702), $t = 5$, $y = 656$. Obviously this pair we found ((2662; 656)) is a palindrome solution of the equation $x - y = 2006$.

8. a) [2] Proof: let's assume that there exists infinitely increasing arithmetic sequence containing only palindromic numbers: $a_{n+1} = a_n + d$, $a_n \in \boldsymbol{P}$ ($n \in \boldsymbol{N}$), $d \in \boldsymbol{N}$. Since the sequence is infinitely increasing, all its elements (except probably several first elements) have number of digits greater than the number of digits in d. For each of these elements, the following element either has the same number of digits as preceding one, or it has exactly one extra digit (note that we are not comparing digits, we are comparing numbers of digits). Since the sequence is infinitely increasing, there exists $n \in \boldsymbol{N}$ such that the number of digits in a_n is greater than the number of digits in d, and the number of digits in a_{n+1} is equal to the number of digits in a_n increased by 1. Therefore the a_{n+1}'s left-most (and right-most) digit is 1, and the a_n's left-most (and right-most) digit is 9. So the d's right-most digit is 2 ($a_{n+1} = a_n + d$), the a_{n+2}'s right-most (and left-most) digit is 3 ($a_{n+2} = a_{n+1} + d$), but if we add number a_{n+1} that starts from 1, and d, that has less number of digits than a_{n+1}, we cannot obtain a_{n+2} that starts from 3! This contradiction proves that every infinitely increasing arithmetic sequence contains at least one element that is not a palindromic number.

b) [2] Proof: let's assume that there exists infinitely increasing Fibonacci-like sequence containing only palindromic numbers: $a_{n+2} = a_{n+1} + a_n$, $a_n \in \boldsymbol{P}$ ($n \in \boldsymbol{N}$). Since the sequence is increasing, $a_n < a_{n+1}$, $a_{n+1} < a_{n+2} = a_{n+1} + a_n < 2a_{n+1}$. Therefore, for each of sequence elements (except probably the first one), the following element either has the same number of digits as preceding one, or it has exactly one extra digit (note that we are not comparing digits, we are comparing numbers of digits). Since the sequence is infinitely increasing, there exists $n \in \boldsymbol{N}$ such that the number of digits in a_{n+2} is equal to the number of digits in a_{n+1} increased by 1. Let k be the number of digits in a_{n+1}, then $k + 1$ is the number of digits in a_{n+2}, and $10^{k-1} \leq a_{n+1} < 10^k$, $10^k \leq a_{n+2} < 2a_{n+1} < 2 \cdot 10^k$, therefore the a_{n+2}'s left-most (and right-most) digit is 1. Since $10^k \leq a_{n+2} < a_{n+3} < 2a_{n+2} < 4 \cdot 10^k$, the a_{n+3}'s left-most (and right-most) digit is either 1 or 2 or 3. If the a_{n+3}'s right-most digit is 1, the a_{n+2}'s right-most digit is 1, so the a_{n+1}'s right-most digit is 0, and a_{n+1} is not a palindrome. If the a_{n+3}'s right-most digit is 2 (3), the a_{n+2}'s right-most digit is 1, so the a_{n+1}'s right-most (and left-most) digit is 1 (2), in this case the number $2a_{n+1}$ has the same number of digits (k) as a_{n+1}, and therefore is less than a_{n+2} that has $k + 1$ digits. This contradiction proves that every infinitely increasing Fibonacci-like sequence contains at least one element that is not a palindromic number.

9. [4] Proof: we will prove that for every $n \in \boldsymbol{N}$ the $(n + 1)$-digit number $a_n = 2\cdot 10^n + 1$ could not be represented as a sum of two palindromes. Let's assume that there are $x, y \in \boldsymbol{P}$: $x + y = a_n$. WLOG we can assume that $x \geq y \geq 1$. Therefore $x \leq 2\cdot 10^n$, $x \geq a_n/2 = 10^n + 1/2$. But x cannot be equal to $2\cdot 10^n$ (x is a palindrome), so $x < 2\cdot 10^n$, and the x's left-most digit is 1. Since x is a palindrome, its right-most digit is also 1, and $y = a_n - x$ is divisible by 10. Therefore, y is not a palindrome. This contradiction proves that for every $n \in \boldsymbol{N}$ the number $a_n = 2\cdot 10^n + 1$ could not be represented as a sum of two palindromes. Since all numbers a_n ($n \in \boldsymbol{N}$) are distinct, there are infinitely many of them.

10. [4] Proof: let n be a k-digit natural number, $k \geq 2$, $m = [(k + 1)/2]$, $m \leq (k + 1)/2 \leq m + 1/2$, $2m - 1 \leq k \leq 2m$, $m \in \boldsymbol{N}$. Therefore $10^{2m-2} \leq 10^{k-1}$ (the least k-digit natural number) $\leq n \leq 10^k - 1$ (the greatest k-digit natural number) $\leq 10^{2m} - 1$, and $P(n) \leq P(10^{2m} - 1) = |\boldsymbol{P}_1| + \ldots |\boldsymbol{P}_{2m}| =$ (see answer for the Problem 1a) $9 + 9 + 90 + 90 + \ldots + 9\cdot 10^{m-1} + 9\cdot 10^{m-1} =$ $= 2(9 + 90 + 9\cdot 10^{m-1}) = 2(10^m - 1) < 2\cdot 10^m$. Finally we can bound $\frac{P(n)}{n}$: $0 \leq \frac{P(n)}{n} \leq \frac{2\cdot 10^m}{10^{2m-2}} = \frac{2}{10^{m-2}}$. When $n \to +\infty$, $k \to +\infty$, $m \to +\infty$, $10^{m-2} \to +\infty$, $2\cdot 10^{2-m} \to 0$, and therefore $\frac{P(n)}{n} \to 0$.