Lagrange's Four Square Theorem

Mr. Sterr
NYC Math Team

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Warmup Problems

1. If 23 is written as the sum of the squares of 4 positive integers (not necessarily different), what is the largest square in this sum? *NYML 2015 2-1*

2. How many ways can you express 73 as the sum of the squares of 4 positive integers?

3. Are the following statements true or false? Justify your answer.
   - "All positive integers can be written as the sum of the squares of 2 integers."
   - "All positive integers can be written as the sum of the squares of 3 integers."

Lagrange's Four Squares Theorem

4. Verify the theorem for the first ten integers

5. Expand the following expression and factor the result:
   $(ae+bf+cg+dh)^2+(af-be+ch-dg)^2+(ag-bh-ce+df)^2+(ah+bg-cf-de)^2$

Theorem 1
If m and n are positive integers that are each the sum of four squares, then their product mn is also the sum of four squares.

Euler's Four-Square Identity

6. Use the identity we just proved to show that 70 can be written as a sum of 4 squares.

Lemma 1
If $2m$ is a sum of two squares, then so is m.

7. Prove Lemma 1

Lemma 2
If p is an odd prime, then there exists an integer $k<p$ such that kp can be written as the sum of four squares of integers.

Theorem 2
Any prime p can be written as the sum of four squares of integers.
There exist integers a,b,c,d such that $p=a^2+b^2+c^2+d^2$.