

The numbers in square brackets represent the point value for each item.

ADDITION AND ANGLES OF TRIANGLES

Definitions

For any positive a , b , and c satisfying the following condition: $0 < a \leq b \leq c < a + b$ (*), we define $T(a, b, c)$ as a triangle with sides a , b , and c .

For any triangles $T(a_1, b_1, c_1)$ and $T(a_2, b_2, c_2)$ we define their **sum** $T(a_1, b_1, c_1) + T(a_2, b_2, c_2)$ as triangle $T(a_1 + a_2, b_1 + b_2, c_1 + c_2)$.

Problems

- Prove that triangle $T(a, b, c)$ is acute (right, obtuse) if and only if $a^2 + b^2 - c^2 > 0$ ($= 0, < 0$) respectively. [2]
 - Prove that for every triangle $T(a, b, c)$ there exists an acute triangle $T(\sqrt{a}, \sqrt{b}, \sqrt{c})$. [2]
- Prove the correctness of the above definition of the sum of triangles, i.e. prove that for every two triangles $T(a_1, b_1, c_1)$ and $T(a_2, b_2, c_2)$ numbers $a_1 + a_2$, $b_1 + b_2$, and $c_1 + c_2$ satisfy the condition (*). [2]
 - Find an explicit error in the following “proof” without referring to any subsequent problem:
Statement: The sum of two obtuse triangles could be an acute triangle.
Proof: A triangle with sides 2, 3, 4 exists ($0 < 2 \leq 3 \leq 4 < 2 + 3$) and is an obtuse one ($2^2 + 3^2 - 4^2 = -3 < 0$, see the Problem 1). A triangle with sides 6, 6, 6 exists and is an equilateral one. An equilateral triangle is also an acute one (well-known fact, see also the Problem 1). Finally, the acute triangle $T(6, 6, 6)$ is the sum of two obtuse triangles: $T(2, 3, 4)$ and $T(4, 3, 2)$. [2]
- Prove that the sum of two right triangles is never an acute triangle. [2]
 - Prove that the sum of two obtuse triangles is always an obtuse triangle. [2]
- Provide an example that shows that the sum of a right and an acute triangles could be a right triangle. [1]
 - Provide an example that shows that the sum of two acute triangles could be a right triangle. [1]
 - Provide an example that shows that the sum of two right triangles could be a right triangle. [1]
 - Provide an example that shows that the sum of two acute triangles could be an obtuse triangle. [1]
- Prove that the sum of an acute and an equilateral triangles is always an acute triangle. [1]
 - Prove that the sum of a right and an equilateral triangles is always an acute triangle. [1]
 - Prove that for every obtuse triangle there exists an equilateral triangle such that their sum is a right triangle. [2]
- Prove that if the sum of two triangles is an isosceles triangle then both original triangles are isosceles. [2]
 - Prove that if the sum of two triangles is an equilateral triangle then both original triangles are equilateral. [1]
 - Is it true that the sum of two isosceles triangles is always an isosceles triangle? Justify your answer. [1]
- A triangle is represented as the sum of two triangles. Prove that if at least two of these three triangles are similar to each other then all three triangles are similar to each other. [2]
 - A right triangle is represented as the sum of two right triangles. Prove that all three right triangles are similar to each other. [2]
- Prove that an equilateral triangle cannot be represented as the sum of two triangles not similar to each other. [1]
 - Prove that every non-equilateral triangle could be represented as the sum of two triangles not similar to each other. [3]
- Provide an example of a triangle that is represented as the sum of two other triangles, and the greatest angle of the first triangle is greater than the greatest angle of the second and third triangles. [1]
 - A triangle is represented as the sum of another triangle and an equilateral triangle. Prove that the greatest angle of the first triangle is not greater than the greatest angle of the second and third triangles. [3]
- A triangle is represented as the sum of two other triangles, and the greatest angles for each of these three triangles are the same. Prove that all three triangles are similar to each other. [4]

The numbers in square brackets represent the point value for each item.

ADDITION AND ANGLES OF TRIANGLES

Answers and Solutions

1. a) [2] A type of a triangle $T(a, b, c)$ (acute, right, obtuse) depends on its greatest angle (γ) that is opposite to its greatest side (c), namely a triangle is acute (right, obtuse) if and only if its greatest angle is less than (equal to, greater than) $\pi/2$. Since the triangle's greatest angle is always between 0 and π , the above conditions $\gamma < \pi/2$ ($= \pi/2, > \pi/2$) are equivalent to $\cos \gamma > 0$ ($= 0, < 0$). Since $\cos \gamma = \frac{a^2 + b^2 - c^2}{2ab}$ ($a > 0, b > 0, c > 0$), they are further equivalent to $a^2 + b^2 - c^2 > 0$ ($= 0, < 0$) respectively.

b) [2] By definition of $T(a, b, c)$, numbers a, b , and c satisfy the condition (*): $0 < a \leq b \leq c < a + b$. Let's consider numbers $a_1 = \sqrt{a}, b_1 = \sqrt{b}, c_1 = \sqrt{c}$. It is obvious that $0 < a_1 \leq b_1 \leq c_1$. To prove existence of $T(a_1, b_1, c_1)$, we just need to prove that $c_1 < a_1 + b_1$. Indeed, $c_1 = \sqrt{c} < \sqrt{a+b} = \sqrt{a_1^2 + b_1^2} < \sqrt{a_1^2 + 2a_1b_1 + b_1^2} = \sqrt{(a_1 + b_1)^2} = a_1 + b_1$. Since $a_1^2 + b_1^2 - c_1^2 = a + b - c < 0$, $T(a_1, b_1, c_1)$ is an acute triangle (according to the Problem 1a).

2. a) [2] Let's consider two triangles $T(a_1, b_1, c_1)$ and $T(a_2, b_2, c_2)$. Numbers a_1, b_1, c_1 , and a_2, b_2, c_2 satisfy the condition (*): $0 < a_1 \leq b_1 \leq c_1 < a_1 + b_1$, $0 < a_2 \leq b_2 \leq c_2 < a_2 + b_2$. By adding these inequalities we prove that numbers $a_1 + a_2, b_1 + b_2$, and $c_1 + c_2$ also satisfy the condition (*):

$$0 < a_1 + a_2 \leq b_1 + b_2 \leq c_1 + c_2 < (a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1 + b_2).$$

b) [2] $T(4, 3, 2)$ is not defined! $T(a, b, c)$ is only defined for numbers a, b , and c that satisfy condition (*).

3. a) [2] Let's consider two right triangles $T(a_1, b_1, c_1)$ and $T(a_2, b_2, c_2)$, $c_1^2 = a_1^2 + b_1^2, c_2^2 = a_2^2 + b_2^2$ (according to the Problem 1a). Let's assume that the sum of these two right triangles is an acute triangle $T(a_1 + a_2, b_1 + b_2, c_1 + c_2)$. According to the Problem 1a, $(c_1 + c_2)^2 < (a_1 + a_2)^2 + (b_1 + b_2)^2$. Subtracting the above equalities, we obtain $0 < c_1c_2 < a_1a_2 + b_1b_2$. Squaring both sides we obtain $(a_1^2 + b_1^2)(a_2^2 + b_2^2) = c_1^2c_2^2 < (a_1a_2 + b_1b_2)^2, a_1^2b_2^2 + a_2^2b_1^2 < 2a_1a_2b_1b_2, (a_1b_2 - a_2b_1)^2 < 0$. This contradiction shows that our original assumption (the sum of these two right triangles is an acute triangle) was wrong.

b) [2] Let's consider two obtuse triangles $T(a_1, b_1, c_1)$ and $T(a_2, b_2, c_2)$, $c_1^2 > a_1^2 + b_1^2, c_2^2 > a_2^2 + b_2^2$ (according to the Problem 1a). Let's assume that the sum of these two obtuse triangles is a non-obtuse triangle $T(a_1 + a_2, b_1 + b_2, c_1 + c_2)$. According to the Problem 1a, $(c_1 + c_2)^2 \leq (a_1 + a_2)^2 + (b_1 + b_2)^2$. Subtracting the above inequalities, we obtain $0 < c_1c_2 < a_1a_2 + b_1b_2$. Squaring both sides we obtain $(a_1^2 + b_1^2)(a_2^2 + b_2^2) < c_1^2c_2^2 < (a_1a_2 + b_1b_2)^2, a_1^2b_2^2 + a_2^2b_1^2 < 2a_1a_2b_1b_2, (a_1b_2 - a_2b_1)^2 < 0$. This contradiction shows that our original assumption (the sum of these two obtuse triangles is a non-obtuse triangle) was wrong.

4. a) [1] $T(3, 4, 5) + T(3, 8, 8) = T(5, 12, 13)$, $0 < 3 \leq 4 \leq 5 < 3 + 4, 3^2 + 4^2 = 5^2, 0 < 3 \leq 8 \leq 8 < 3 + 8, 3^2 + 8^2 > 8^2, 0 < 5 \leq 12 \leq 13 < 5 + 12, 5^2 + 12^2 = 13^2$. According to the Problem 1a, $T(3, 4, 5)$ and $T(5, 12, 13)$ are right triangles, and $T(3, 8, 8)$ is an acute one.

b) [1] $T(3, 4, 4.9) + T(3, 8, 8.1) = T(5, 12, 13)$, $0 < 3 \leq 4 \leq 4.9 < 3 + 4, 3^2 + 4^2 > 4.9^2, 0 < 3 \leq 8 \leq 8.1 < 3 + 8, 3^2 + 8^2 > 8.1^2, 0 < 5 \leq 12 \leq 13 < 5 + 12, 5^2 + 12^2 = 13^2$. According to the Problem 1a, $T(3, 4, 4.9)$ and $T(3, 8, 8.1)$ are acute triangles, and $T(5, 12, 13)$ is a right one.

c) [1] $T(3, 4, 5) + T(6, 8, 10) = T(9, 12, 15)$, $0 < 3 \leq 4 \leq 5 < 3 + 4, 3^2 + 4^2 = 5^2, 0 < 6 \leq 8 \leq 10 < 6 + 8, 6^2 + 8^2 = 10^2, 0 < 9 \leq 12 \leq 15 < 9 + 12, 9^2 + 12^2 = 15^2$. According to the Problem 1a, $T(3, 4, 5)$, $T(6, 8, 10)$, and $T(9, 12, 15)$ are right triangles.

d) [1] $T(3, 4, 4.95) + T(5, 12, 12.95) = T(8, 16, 17.9)$, $0 < 3 \leq 4 \leq 4.95 < 3 + 4, 3^2 + 4^2 > 4.95^2, 0 < 5 \leq 12 \leq 12.95 < 5 + 12, 5^2 + 12^2 > 12.95^2, 0 < 8 \leq 16 \leq 17.9 < 8 + 16, 8^2 + 16^2 < 17.9^2$. According to the Problem 1a, $T(3, 4, 4.95)$ and $T(5, 12, 12.95)$ are acute triangles, and $T(8, 16, 17.9)$ is an obtuse one.

5. a) [1] Let's consider an acute triangle $T(a, b, c)$, $c^2 < a^2 + b^2$ (according to the Problem 1a). For any equilateral triangle $T(x, x, x)$ we obtain $(a+x)^2 + (b+x)^2 - (c+x)^2 = (a^2 + b^2 - c^2) + x^2 + 2x(a+b-c) > 0$. According to the Problem 1a, triangle $T(a, b, c) + T(x, x, x)$ is an acute one.

b) [1] Let's consider a right triangle $T(a, b, c)$, $c^2 = a^2 + b^2$ (according to the Problem 1a). For any equilateral triangle $T(x, x, x)$ we obtain $(a+x)^2 + (b+x)^2 - (c+x)^2 = (a^2 + b^2 - c^2) + x^2 + 2x(a+b-c) > 0$. According to the Problem 1a, triangle $T(a, b, c) + T(x, x, x)$ is an acute one.

c) [2] Let's consider an obtuse triangle $T(a, b, c)$, $c^2 > a^2 + b^2$ (according to the Problem 1a). We want the sum of this triangle and an equilateral triangle $T(x, x, x)$ to be a right triangle: $(a+x)^2 + (b+x)^2 - (c+x)^2 = (a^2 + b^2 - c^2) + x^2 + 2x(a+b-c) = 0$. The previous condition is equivalent to $(x+d)^2 = d^2 + (c^2 - a^2 - b^2) > d^2$, where $d = (a+b-c) > 0$, and this equation has a positive solution $x = \sqrt{d^2 + (c^2 - a^2 - b^2)} - d > 0$.

6. a) [2] Let's consider two triangles $T(a_1, b_1, c_1)$ and $T(a_2, b_2, c_2)$, $0 < a_1 \leq b_1 \leq c_1$, $0 < a_2 \leq b_2 \leq c_2$, and the sum of these two triangles is an isosceles triangle $T(a_1 + a_2, b_1 + b_2, c_1 + c_2)$. Since $0 < a_1 + a_2 \leq b_1 + b_2 \leq c_1 + c_2$, $T(a_1 + a_2, b_1 + b_2, c_1 + c_2)$ is an isosceles triangle if and only if $a_1 + a_2 = b_1 + b_2$ or $b_1 + b_2 = c_1 + c_2$. Let's consider the first condition (similar reasoning could be applied to the second one). We have $a_1 \leq b_1$, $a_2 \leq b_2$, and $a_1 + a_2 = b_1 + b_2$. It implies $a_1 = b_1$ and $a_2 = b_2$ (otherwise $a_1 + a_2 < b_1 + b_2$), so both triangles $T(a_1, b_1, c_1)$ and $T(a_2, b_2, c_2)$ are isosceles ones.

b) [1] Let's consider two triangles $T(a_1, b_1, c_1)$ and $T(a_2, b_2, c_2)$, $0 < a_1 \leq b_1 \leq c_1$, $0 < a_2 \leq b_2 \leq c_2$, and the sum of these two triangles is an equilateral triangle $T(a_1 + a_2, b_1 + b_2, c_1 + c_2)$, $a_1 + a_2 = b_1 + b_2 = c_1 + c_2$. We have $a_1 \leq b_1 \leq c_1$, $a_2 \leq b_2 \leq c_2$, and $a_1 + a_2 = b_1 + b_2 = c_1 + c_2$. It implies $a_1 = b_1 = c_1$ and $a_2 = b_2 = c_2$ (otherwise $a_1 + a_2 < b_1 + b_2$ or $b_1 + b_2 < c_1 + c_2$), so both triangles $T(a_1, b_1, c_1)$ and $T(a_2, b_2, c_2)$ are equilateral ones.

c) [1] The sum of two isosceles triangles is not always an isosceles triangle: $T(2, 2, 3) + T(2, 3, 3) = T(4, 5, 6)$, triangles $T(2, 2, 3)$ and $T(2, 3, 3)$ are isosceles, and triangle $T(4, 5, 6)$ is not isosceles.

7. a) [2] Let's note that $T(a_1, b_1, c_1) \sim T(a_2, b_2, c_2)$ if and only if $\frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{c_2}{c_1}$ (the shortest sides are the

corresponding ones, the longest sides are the corresponding ones, and the middle sides are the corresponding ones). Let's consider three triangles: $T(a_1, b_1, c_1)$, $T(a_2, b_2, c_2)$, and their sum $T(a_1 + a_2, b_1 + b_2, c_1 + c_2)$; and two of these two triangles are similar to each other. There are logically two cases: the first two triangles are similar to each other, or one of them is similar to the third triangle. If $T(a_1, b_1, c_1) \sim T(a_2, b_2, c_2)$, then $\frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{c_2}{c_1}$, $\frac{a_2 + a_1}{a_1} = \frac{b_2 + b_1}{b_1} = \frac{c_2 + c_1}{c_1}$,

i.e. the third triangle is similar to the first one, and all three triangles are similar to each other. If $T(a_1, b_1, c_1) \sim T(a_1 + a_2, b_1 + b_2, c_1 + c_2)$, then $\frac{a_2 + a_1}{a_1} = \frac{b_2 + b_1}{b_1} = \frac{c_2 + c_1}{c_1}$, $\frac{a_2}{a_1} + 1 = \frac{b_2}{b_1} + 1 = \frac{c_2}{c_1} + 1$, $\frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{c_2}{c_1}$, i.e. the second triangle is similar to the first one, and all three triangles are similar to each other.

b) [2] Let's note that $T(a_1, b_1, c_1) \sim T(a_2, b_2, c_2)$ if and only if $\frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{c_2}{c_1}$ (the shortest sides are the

corresponding ones, the longest sides are the corresponding ones, and the middle sides are the corresponding ones). Let's consider three right triangles: $T(a_1, b_1, c_1)$, $T(a_2, b_2, c_2)$, and their sum $T(a_1 + a_2, b_1 + b_2, c_1 + c_2)$. According to the Problem 1a, $c_1^2 = a_1^2 + b_1^2$, $c_2^2 = a_2^2 + b_2^2$, and $(c_1 + c_2)^2 = (a_1 + a_2)^2 + (b_1 + b_2)^2$. Subtracting the above equalities, we obtain $0 < c_1 c_2 = a_1 a_2 + b_1 b_2$. Squaring both sides we obtain $(a_1^2 + b_1^2)(a_2^2 + b_2^2) = c_1^2 c_2^2 = (a_1 a_2 + b_1 b_2)^2$,

$a_1^2 b_2^2 + a_2^2 b_1^2 = 2a_1 a_2 b_1 b_2$, $(a_1 b_2 - a_2 b_1)^2 = 0$, $\frac{a_2}{a_1} = \frac{b_2}{b_1}$. Since both triangles $T(a_1, b_1, c_1)$ and $T(a_2, b_2, c_2)$ are right ones with

hypotenuses c_1 and c_2 respectively, they are similar to each other. According to the Problem 7a, all three triangles are similar to each other.

8. a) [1] If an equilateral triangle is represented as the sum of two triangles, these two triangles are also equilateral (according to the Problem 6b). But all equilateral triangles are similar to each other (all their angles are equal), therefore an equilateral triangle cannot be represented as the sum of two triangles not similar to each other.

b) [3] Let's consider a non-equilateral triangle $T(a, b, c)$. Numbers a, b , and c satisfy the condition (*): $0 < a \leq b \leq c < a + b$, and also $c > a$ since this triangle is not an equilateral one. Let's select $x > 0, x < a, x < (a + b - c)$, and consider "triangle" $T(a - x, b - x, c - x)$ and equilateral triangle $T(x, x, x)$. Triangle $T(a - x, b - x, c - x)$ exists since numbers $a - x, b - x$, and $c - x$ satisfy the condition (*): $0 < a - x \leq b - x \leq c - x < (a + b - x) - x = (a - x) + (b - x)$. Triangle $T(a - x, b - x, c - x)$ is not an equilateral one since $a - x < c - x$, and therefore it is not similar to an equilateral triangle $T(x, x, x)$. Finally, an arbitrary non-equilateral triangle $T(a, b, c)$ is represented as the sum of two triangles $T(a - x, b - x, c - x)$ and $T(x, x, x)$ that are not similar to each other.

9. a) [1] We can take an example from the Problem 4d in which the sum of two acute triangles is an obtuse triangle. The greatest angle of two acute triangles is less than $\pi/2$, and the greatest angle of an obtuse triangle is greater than $\pi/2$, so the greatest angle of the sum of these two triangles is greater than the greatest angle of both triangles.

b) [3] Let's consider a triangle $T(a, b, c)$ and an equilateral triangle $T(x, x, x)$. The first triangle's greatest angle (γ) is opposite to its greatest side (c). Let's also consider their sum $T(a + x, b + x, c + x)$, its greatest angle is opposite to its greatest side ($c + x$). We want to prove that the greatest angle of $T(a + x, b + x, c + x)$ is not greater than the greatest angle of $T(a, b, c)$ and $T(x, x, x)$. The greatest angle of $T(x, x, x)$ is $\pi/3$, and the greatest angle of $T(a, b, c)$ is at least $\pi/3$ (otherwise all its angles would be less than $\pi/3$, and their sum (π) would be less than $\pi/3 + \pi/3 + \pi/3 = \pi$). Therefore, the greatest angle of $T(a, b, c)$ and $T(x, x, x)$ always has the same value as the greatest angle of $T(a, b, c)$, and we want to prove that the greatest

angle of $T(a + x, b + x, c + x)$ is not greater than the greatest angle of $T(a, b, c)$. $\cos \gamma = \frac{a^2 + b^2 - c^2}{2ab}$, the triangle's greatest

angle is always between 0 and π , and function $y = \cos x$ is decreasing on this interval. Therefore the above condition is

equivalent to $\frac{a^2 + b^2 - c^2}{2ab} \leq \frac{(a + x)^2 + (b + x)^2 - (c + x)^2}{2(a + x)(b + x)}$,

$$(a^2 + b^2 - c^2)(a + x)(b + x) \leq ((a + x)^2 + (b + x)^2 - (c + x)^2)ab,$$

$$x^2(c^2 - a^2 - b^2 + ab) + x(2a^2b + 2ab^2 - 2abc - (a + b)(a^2 + b^2 - c^2)) \geq 0,$$

$$x^2(c^2 - a^2 - b^2 + ab) + x(a^2b + ab^2 - 2abc - a^3 - b^3 + ac^2 + bc^2) \geq 0,$$

$$c^2 - a^2 - b^2 + ab = (c - b)(c + b) + a(b - a) \geq 0,$$

$$a^2b + ab^2 - 2abc - a^3 - b^3 + ac^2 + bc^2 = (c^2(a + b) - 2abc - b^2(a + b) + 2ab^2) + (a^2b - a^3) =$$

$$= (a + b)(c^2 - b^2) - 2ab(c - b) + a^2(b - a) = (c - b)((a + b)(c + b) - 2ab) + a^2(b - a) =$$

$$= (c - b)((a + b)(c + b) - (a + 0)(b + b)) + a^2(b - a) \geq 0. \text{ Since } 0 < a \leq b \leq c, x > 0, \text{ all the inequalities above are true.}$$

10. [4] Let's consider three triangles $T(a_1, b_1, c_1)$, $T(a_2, b_2, c_2)$, and $T(a_3, b_3, c_3)$. Let the third triangle be represented as the sum of the first two triangles ($a_3 = a_1 + a_2, b_3 = b_1 + b_2, c_3 = c_1 + c_2$), and all these three triangles have the same greatest angles (i.e. angles that are opposite to their longest sides) with value γ . Let's draw angle AOB with value γ , and put points A_1, A_2 , and A_3 on $[OA)$ and B_1, B_2 , and B_3 on $[OB)$: $|OA_1| = a_1, |OA_2| = a_2, |OA_3| = a_3, |OB_1| = b_1, |OB_2| = b_2, |OB_3| = b_3$, therefore $|A_1B_1| = c_1, |A_2B_2| = c_2, |A_3B_3| = c_3$. From the conditions above we obtain

$\overrightarrow{OA_3} = \overrightarrow{OA_1} + \overrightarrow{OA_2}, \overrightarrow{OB_3} = \overrightarrow{OB_1} + \overrightarrow{OB_2}, \overrightarrow{OA_3} = \overrightarrow{OA_1} + \overrightarrow{OA_2}, \overrightarrow{OB_3} = \overrightarrow{OB_1} + \overrightarrow{OB_2}$, and therefore

$\overrightarrow{A_3B_3} = \overrightarrow{OB_3} - \overrightarrow{OA_3} = (\overrightarrow{OB_1} + \overrightarrow{OB_2}) - (\overrightarrow{OA_1} + \overrightarrow{OA_2}) = (\overrightarrow{OB_1} - \overrightarrow{OA_1}) + (\overrightarrow{OB_2} - \overrightarrow{OA_2}) = \overrightarrow{A_1B_1} + \overrightarrow{A_2B_2}$. From the other side,

$|\overrightarrow{A_3B_3}| = c_3 = c_1 + c_2 = |\overrightarrow{A_1B_1}| + |\overrightarrow{A_2B_2}|$, so vectors $\overrightarrow{A_1B_1}, \overrightarrow{A_2B_2}$ and $\overrightarrow{A_3B_3}$ are collinear, and therefore triangles A_1OB_1 ($T(a_1, b_1, c_1)$), A_2OB_2 ($T(a_2, b_2, c_2)$), and A_3OB_3 ($T(a_3, b_3, c_3)$) are similar to each other.