New York City Math Team **Mr. Jaishankar**

Combinatorics

Warmup:

i.) Review some problems from assignment given at the last practice
ii.) Prove $\binom{n}{k} = \binom{n}{n-k}$ and $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ using an algebraic argument
iii.) Prove the above identities using a combinatorial argument

Introductory Problem:

How many non-negative integer solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 < 10$?

Class Examples:

1.) Let S be the set $(3, 4, 4, 5, 5, 6, 7, 7, 7)$. How many numbers can be obtained as the product of two or more of the numbers in S?

2.) Given 8 distinguishable rings, let n be the number of possible 5-ring arrangements on the 4 fingers (not the thumb) of one hand. The order of rings on each finger is significant, but it is not required that each finger have a ring. Find the leftmost three non-zero digits of n.

3.) Let S be the set $(3^0, 3^1, 3^2, 3^3, 3^4)$. Compute the number of positive integers less than or equal to 100 that can be expressed as a sum of 3 or fewer members of S if we are allowed to use the same power more than once.
Ex: $5 = 3^0 + 3^0 + 3^1, 10 = 3^2 + 3^1, 13 = 3^2 + 3^1 + 3^0$.
Note that we cannot express the number 8 in such a way.

Homework:

4.) Let n be the number of ordered quadruples (x_1, x_2, x_3, x_4) of positive odd integers such that

$$\sum_{k=1}^{4} x_k = 98$$

Compute $\frac{n}{100}$

5.) The probability that a set of three distinct vertices chosen at random from among the vertices of a regular n-gon determine an obtuse triangle is $\frac{93}{125}$. Compute the sum of all values of n.

6.) There are 5 identical green marbles and a large identical supply of red marbles. Some of the red marbles and the green ones are arranged in a row and finds that the number of marbles whose right hand neighbor is the same color as themselves is equal to the number of marbles whose right hand neighbor is the other color. An example of such an arrangement is $GGRRRGGRG$. Let m be the maximum number of red marbles for which such an arrangement is possible, and N be the number of ways the $m+5$ marbles can be arranged to satisfy the requirement. Compute N.

7.) In a sequence of coin tosses, one can keep a record of the number of instances when a tail is immediately followed by a head, a head is immediately followed a head, etc. Denote the instances as TH, HH, etc. How many different sequences of 15 coin tosses will contain exactly 2 HH, 3 HT, 4 TH and 5 TT subsequences?

Example: There are **exactly** 5 HH's, 3 HT's, 2 TH's and 4 TT's in the 15-toss sequence

$$HHTTHHHHTHHTTTT$$

8.) Let S be the number of ordered sets of positive integers $(a_1, a_2, a_3, a_4, a_5, a_6)$ such that:

i.) $a_k \geq k, k = 1, 2, \ldots 6$
ii.) $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \leq 100$.

S can be written in the form $\binom{n}{k}$. Compute all possible values of $n + k$