Combinatorics

Techniques

1. Basics: Choose k out of n. Use a bijection to establish two sets are the same size. Arithmetic operations are fundamentally counting tools.
2. $|A \cup B| = |A| + |B| - |A \cap B|$: If I roll two dice, what's the probability that at least one of them is a 1?
3. How many ways are there to arrange the letters in the word "banana"? Use chooses or a division argument.
4. "Stars and Bars": How many ways are there to give 5 cookies to 3 children? What if each child must receive at least one cookie?

AIME Problems:

Examples

1. 1989/2: Ten points are marked on a circle. How many distinct convex polygons of three or more sides can be drawn using some (or all) of the ten points as vertices?
2. 2004 II/2: A jar has 10 red candies and 10 blue candies. Terry picks two candies at random, then Mary picks two of the remaining candies at random. Find the probability that they get the same color combination, irrespective of order.
3. 1990/8: In a shooting match, eight clay targets are arranged in two hanging columns of three targets each and one column of two targets. A marksman is to break all the targets according to the following rules: 1) The marksman first chooses a column from which a target is to be broken. 2) The marksman must then break the lowest remaining target in the chosen column. If the rules are followed, in how many different orders can the eight targets be broken?
4. 1995/3: Starting at (0,0), an object moves in the coordinate plane via a sequence of steps, each of length one. Each step is left, right, up, or down, all four equally likely. Find the probability that the object reaches (2,2) in six or fewer steps.
5. 2001 I/6: A fair die is rolled four times. Find the probability that each of the final three rolls is at least as large as the roll preceding it.
6. 2009 I/9. A game show offers a contestant three prizes A, B and C, each of which is worth a whole number of dollars from $1 to $9999 inclusive. The contestant wins the prizes by correctly guessing the price of each prize in the order A, B, C. As a hint, the digits of the three prices are given. On a particular day, the digits given were 1, 1, 1, 1, 3, 3, 3. Find the number of possible guesses for all three prizes consistent with the hint.

Miscellaneous Problems

7. 1990/9: A fair coin is to be tossed 10 times. Find the probability that heads never occur on consecutive tosses.
8. 1996/6: In a five-team tournament, each team plays one game with every other team. Each team has a 50% chance of winning any game it plays. (There are no ties.) Find the probability that the tournament will produce neither an undefeated team nor a winless team.
9. 2008 I/9: Ten identical crates each of dimensions 3 ft x 4 ft x 6 ft. The first crate is placed flat on the floor. Each of the remaining nine crates is placed, in turn, flat on top of the previous crate, and the orientation of each crate is chosen at random. Find the probability that the stack of crates is exactly 41 ft tall.
10. 1999/10: Ten points in the plane are given, with no three collinear. Four distinct segments joining pairs of these points are chosen at random, all such segments being equally likely. Find the probability that some three of the segments form a triangle whose vertices are among the ten given points.

Harder Problems

11. 1986/13: In a sequence of coin tosses, one can keep a record of instances in which a tail is immediately followed by a head, a head is immediately followed by a head, and etc. We denote these by TH, HH, and etc. For example, in the sequence TTTHHTHTTTHHTTH of 15 coin tosses we observe that there are two HH, three HT, four TH, and five TT subsequences. How many different sequences of 15 coin tosses will contain exactly two HH, three HT, four TH, and five TT subsequences?
12. 2006 I/11: A collection of 8 cubes consists of one cube with edge-length k for each integer k, 1 <= k <= 8. A tower is to be built using all 8 cubes according to the rules: Any cube may be the bottom cube in the tower. The cube immediately on top of a cube with edge-length k must have edge-length at most k+2. Find the number of different towers that can be constructed.
13. 2010 II/11: Define a T-grid to be a 3 x 3 matrix which satisfies the following two properties. Exactly five of the entries are 1's, and the remaining four entries are 0's. Among the eight rows, columns, and long diagonals, no more than one of the eight has all three entries equal. Find the number of distinct T-grids.
14. 2008 II/12: There are two distinguishable flagpoles, and there are 19 flags, of which 10 are identical blue flags, and 9 are identical green flags. Find the number of distinguishable arrangements using all of the flags in which each flagpole has at least one flag and no two green flags on either pole are adjacent.