NYC Math Team Practice Session
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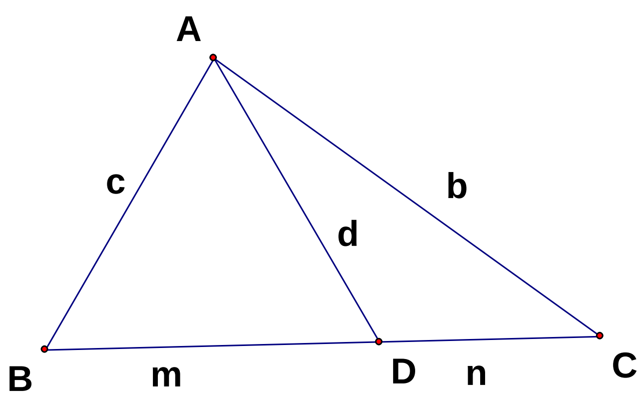
Stewart's Theorem

This is one of the most useful (and most unknown, at least to regular mathematics students) theorems for any fledgling mathematical olympian. (It's a secret to everyone.) I happen to consider this the most useful theorem that they never teach you in geometry, so definitely keep this in mind. But wait—oh no! Mr. Jones and his father have placed an explosive device in the wash basin! Wait a minute…

Part I: Proof the First

1) There is an equation that can allow us to compute the lengths of the sides in the triangle as shown to the left without involving any of the angles! Here, as usual, $BC = a = m + n$, and cevian $AD = d$. Our goal is to derive an equation that will be of use to us. Since we're familiar with right triangles, let's try drawing some. Draw an altitude $\overline{AH}$ in the triangle and let $h = AH$. Let $HD = p$. Use the Pythagorean Theorem to write three equations that relate the variables in the diagram.



2) Use the equations to solve for c^2 in terms of *d*, *m*, and *p*.

3) Use the equations to solve for b^2 in terms of *d*, *n*, and *p*.

4) Great. Now, multiply the equation from problem #2 by *n* and the equation from problem #3 by *m*. Sum the two equations.

5) The end draws near! Try to factor one side of the equation a bit, and see if you can introduce *a*. You should end up proving that $c^2n + b^2m = man + ad^2$.

Part II: Proof the Second

6) Note that this proof is good, but it can be made even better. Let's redo the problem—use the same picture as in the first set. Compute $\cos(\angle ADB)$ in terms of the lengths of the segments in the diagram.

7) Compute $\cos(\angle ADC)$.

8) Use what you know about $\angle ADB$ and $\angle ADC$ to show $c^2n + b^2m = man + ad^2$.

Part III: Problems!

9) Two of the sides of a triangle have length 4 and 9. The median drawn to the third side of this triangle has a length of 6. Compute the length of the third side of this triangle.

10) Prove that for any triangle ΔABC, if $\overline{AD}$ is a median drawn to $\overline{BC}$, then $AB^2 + AC^2 = 2(AD^2 + BD^2)$. (*The result that you have just proved is known as Apollonius' Theorem.*)

11) In ΔABC, X is a point on $\overline{AB}$ such that $AX = 2BX = 4$. If $AC = 7$ and $BC = 5$, compute CX.

12) (Mu Alpha Theta 1987) In ΔADC, $\overline{DB}$ is drawn such that $\overline{DB}$ bisects $\angle ADC$. If $AB = 3$, $AD = 6$, and $CD = 8$, compute DB. (*It is helpful to recall the Angle Bisector Theorem for triangles…*)

13) Using Stewart's Theorem, prove the following useful fact about an angle bisector of a triangle. (*I call it the second half of the Angle Bisector Theorem*.) In the diagram from the front, if $\overline{AD}$ bisects $\angle BAC$, then $d^2 = bc - mn$.

14) (2014 OMO #11) In ΔABC, let Γ be the semicircle inscribed in ΔABC such that its diameter, $\overline{EF}$, lies on $\overline{BC}$, and is tangent to $\overline{AB}$ and $\overline{AC}$. If $BE = 1$, $EF = 24$, and $FC = 3$, compute the perimeter of ΔABC. (*As always, a good diagram is key*.)

15) (ARML 1984 Tiebreaker #1) ΔABC is inscribed in a circle, and P is drawn on the circle such that $\overline{BP}$ bisects $\angle ABC$. If $AB = 6$, $BC = 8$, and $AC = 7$, compute BP. (*To solve this problem, you will need to know a certain theorem in circles. I imagine you could derive it yourself, though…*)

16) (2003 AIME I #7) Let B be a point on $\overline{AC}$ such that $AB = 9$ and $BC = 21$. Let D be a point not on $\overline{AC}$ such that $AD = CD$, and AD and BD are integers. Compute the sum of all possible perimeters of ΔACD. (*Stewart's Theorem is the easy part here. What to do with the rest is up to you…*)

17) Consider a right triangle ABC with legs of length a and b and hypotenuse of length c. Two cevians are drawn from C, intersecting the hypotenuse such that the cevians trisect the hypotenuse. Let p and q be the lengths of the cevians, and x be the length of one of the segments the hypotenuse is divided into. Show that $p^2 + q^2 = 5x^2$.

18) (2013 AMC 10 #23) In ΔABC, $AB = 86$ and $AC = 97$. A circle centered at A and passing through B intersects $\overline{BC}$ at two points: B and X. If BX and CX are integers, compute BC. (*Draw a good diagram here, like always. It also helps here to know how to factor as well…*)