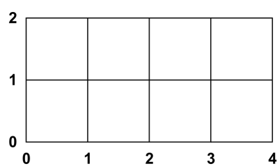
Combinations

The number of different k-element subsets of an n-element set is usually denoted $\binom{n}{k}$ and pronounced "n choose k". It is sometimes written as ${}_nC_k$. The subsets being counted are also known as "combinations". They are different from "permutations" in that order does not matters for combinations.

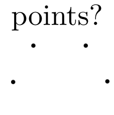
The general formula for combinations is: $\displaystyle\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

The answer to each of the questions below is $\binom{6}{2} = 15$. Your task is to relate each problem to an earlier one in the list, and offer a convincing explanation why the two problems must have the same answer.

1. How many different two-element subsets are there of the set $\{1, 2, 3, 4, 5, 6\}$?
2. How many words (strings of letters) are there, that are made up of four letters "A" and two letters "B"?
3. What is the coefficient of x^2y^4 in the expansion of $(x + y)^6$?
4. How many different paths are there in a grid going from (0, 0) to (4, 2) if each path may only go up or to the right along the grid lines?



5. If in a group of six people everyone shakes everyone else's hand, how many handshakes will there be?
6. How many different line segments are there whose endpoints belong to this set of 6 points?



7. How many words can be made of 5 A's and 2 B's if B's may not be next to each other?
8. Consider the expression __ + __ + __ = 4. How many ways are there to fill in the blanks with non-negative integers and have a correct sum?
9. How many ways are there to distribute 7 apples among 3 people so that each person gets at least one apple?
10. How many four-digit numbers are there consisting of digits 1, 2, and 3, in which digits do not decrease when read from left to right? (E.g. 2233 is such a number, while 2133 is not.)
11. How many ways are there to award 2 concert tickets to 5 people, allowing for both tickets to go to the same person?

Problems on Binomial Coefficients

12. Compute $\binom{6}{0} + \binom{6}{1} + ... + \binom{6}{6}$.
13. Prove that for any integer $n > 0$,

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - ... \pm \binom{n}{n-1} \mp \binom{n}{n} = 0.$$

14. Compute the number of subsets of a set of n elements.
15. Compute the number of subsets of a set of n element, which contain an odd number of elements.
16. Let S be a set of six elements. How many pairs of subsets $T, U \subseteq S$ are there such that $T \subseteq U$?

Problems on Inclusion-Exclusion

17. In a class of music students, 20 are learning to play the piano, and 15 are learning to play the violin. Moreover, 9 of the students are learning to play both the piano and the violin. How many students are learning to play at least one of these two instruments?
18. How many integers in the range 1 to 1000 are divisible by at least one of 2, 3, or 5?
19. How many integers in the range 1 to 1000 are neither perfect squares, nor perfect cubes, nor perfect fourth powers?

More Combinatorics Problems

20. How many diagonals does a decagon (a 10-sided polygon) have?
21. A fair die is rolled four times. The probability that each of the final three rolls is at least as large as the roll preceding it may be expressed in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. Compute the ordered pair (m, n).
22. Prove the following identities using a combinatorial argument:
    (a) $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$
    (b) $\binom{n}{0}^2 + \binom{n}{1}^2 + ... + \binom{n}{n}^2 = \binom{2n}{n}$
23. A team of 10 students decides they all should work in pairs. How many ways are there to divide the 10 students into 5 pairs?
24. How many ways can 10 people form two teams of 5?
25. The n vertices of a polygon are arranged on the circumference of a circle so that no three diagonals intersect in the same point. How many intersection points do these diagonals have? (From "A Decade of the Berkeley Math Circle")
26. How many nine digit positive integers have the property that the digits do not decrease from left to right?

Binomial Theorem

Any power of a *binomial* (i.e. a sum of two terms) may be expanded as follows:

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + ... + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n$$

See the problems for fun applications. Keep in mind that this is an *identity*: it is true for *any* value of x, y. For proof, consider problem (3) from the problems on combinations.

Inclusion-Exclusion Principle

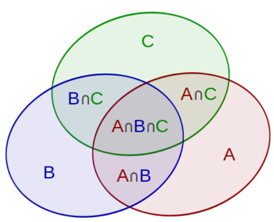
The Principle of Inclusion-Exclusion (or "PIE") provides an organized method to find the size of the union of sets if the sizes of all possible intersections are known.

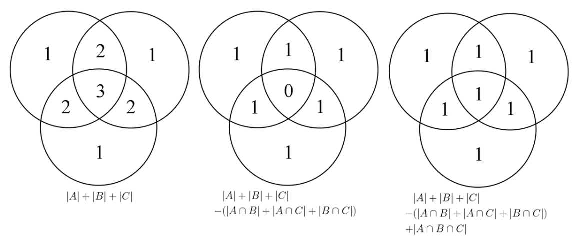
For two sets:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

For three sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$





In general:

$$\left|\bigcup A_i\right| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - ... \pm \sum |A_1 \cap ... \cap A_n|,$$

where all sums are over all possible combinations of that many different subsets.

Proof. We need to show that each element present in *at least* one A_i is counted exactly once on the right-hand side.

If an element a is only in a single A_i, then it is counted in the first sum only, so it is counted once.

If an element a is in two sets, say, A_1 and A_2, then it is counted twice in the first sum (+2), and once in the second (-1), so it is counted once total.

In general, if an element is in k sets, say, $A_1, ..., A_k$, then it is counted k times in the first sum, $\binom{k}{2}$ times in the second sum (once for each pair of sets from $A_1, ..., A_k$), $\binom{k}{3}$ times in the third sum, and so on. In total, it is counted

$$\binom{k}{1} - \binom{k}{2} + \binom{k}{3} + ... \pm \binom{k}{k}$$

times. By problem (13) from the probems on Binomial Expansion, this is equal to $\binom{k}{0} = 1$, so the element is counted exactly once. □

The Derangement Problem

Imagine that you've been signing holiday cards, and have a bunch of signed cards and an addressed envelope for each of them. If you stuff cards into envelopes randomly, what's the probability that *no* card ends up in the right envelope? Surprising, the probability is approximately $1/e$.

A permutation where none of the cards are in the right envelope is called a *derangement*. Let's use $D(n)$ to denote the number of derangements of n elements.

Part 1. Derive the formula for the number of permutations with at least one card in the right envelope using the Inclusion-Exclusion Principle. Use

$$A_i = \{\text{permutations with card } i \text{ in the right envelope}\}$$

1. How many permutations are in A_i? In $A_i \cap A_j$? In $A_i \cap A_j \cap A_k$?
2. How many sets are covered by the sum $\sum |A_i \cap A_j \cap A_k|$?
3. Generalize your result to intersections of k sets, and apply PIE.

With this, $D(n) = n! - \#\{\text{permutations with at least one card in the right envelope}\}$. Simplifying, the formula should come out to:

$$D(n) = n! \cdot \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ... \mp \frac{1}{(n-1)!} \pm \frac{1}{n!}\right). \tag{1}$$

Part 2. It is also interesting to derive a recursive formula for $D(n)$. First ask: where can we put the first element to get a derangement? The answer is: into any of the $n-1$ places after the first one. Let's say the first element ends up in the j^{th} place. What remains are items from 2 through n, and places from 1 through n except j. We can list them like so:

elements:	2	3	...	$j-1$	j	$j+1$	...	n
places:	2	3	...	$j-1$	1	$j+1$	...	n

If the element j goes into position 1, consider carefully how many permutations of the remaining elements produce a derangement? Do you see why it is $D(n-2)$?

Now consider how many permutations produce a derangement if the element j does *not* goes into position 1. Do you see why it is $D(n-1)$?

Since the first element can go into any of $n-1$ places, and for each of these places, the number of suitable permutations of the rest of the elements is $D(n-2) + D(n-1)$, the total number of derangements of n elements is

$$D(n) = (n-1)\big[D(n-2) + D(n-1)\big].$$

Part 3. Use the recursive formula to prove formula (1) by induction.

Part 4. (For those who've taken Calculus) Use the Taylor-series expansion of e^x to prove that the probability of a random permutation being a derangement approaches $\frac{1}{e}$ as n approaches ∞.