1 Team Round

T1. Compute $\dfrac{\sqrt{\sqrt{5}+2}-\sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}-1}}$.

T2. Let k be a positive integer such that the product $2015k$ has exactly 18 divisors. Compute the sum of all possible values of k.

T3. Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}$ and $AB = BC$. A circle with center O on $\overline{CD}$ is tangent to $\overline{DA}$, $\overline{AB}$, and $\overline{BC}$. If $m\angle BCD = 68°$, compute $m\angle CDA$.

T4. If the sum of the cubes of the roots of $x^2 - bx + 10 = 0$ is $6b$, compute the greatest possible value of b.

T5. Compute the maximum value of $\log x + \log y + \log z$ if x, y, and z are positive real numbers that satisfy $x + 4y + 16z = 120$.

T6. A broken automatic card shuffler can only shuffle decks of five cards. Furthermore, it can only perform two types of shuffles. The first type of shuffle only switches the top two cards (leaving the order of the bottom cards unchanged), and the second type of shuffle reverses the order of the bottom four cards (leaving the top card on top). Five cards are numbered with five different numbers and arranged in a random order, with all permutations equally likely. What is the probability that the shuffler cannot put the cards in increasing order, from top to bottom, regardless of the number and types of shuffles it performs?

T7. There are 12 adjacent parking spaces in a parking lot and 8 of them are occupied. A large truck arrives, needing 2 adjacent unoccupied spaces to park. Compute the probability that it will be able to park. (Each arrangement of cars is equally likely.)

T8. If $x + y + z = 7$, $xy + yz + zx = 8$, and $xyz = 2$, compute the maximum value of x.

T9. The three-digit prime number p is written in base 2 as p_2 and in base 5 as p_5, and the two representations share the same last two digits. If the ratio of the number of digits in p_2 to the number of digits in p_5 is 5 to 2, find all possible values of p.

T10. The parabola $y = 3x^2 + 5x - 9$ has focus F. A line passes through F and intersects the parabola at two points, $P_1 = (-4, 19)$ and P_2. Two tangent lines to the parabola are drawn through P_1 and P_2, intersecting at Q. Compute $m\angle P_1QP_2$.

2 Answers

2.1 Team Round

T1-Ans. $\sqrt{2}$

T2-Ans. 623

T3-Ans. 44°

T4-Ans. 6

T5-Ans. 3

T6-Ans. $\frac{9}{10}$

T7-Ans. $\frac{41}{55}$

T8-Ans. $3+\sqrt{7}$

T9-Ans. 601

T10-Ans. 90°

3 Solutions

3.1 Team Round

T1-Sol. $\sqrt{2}$

Square the whole expression to obtain $\dfrac{(\sqrt{5}+2)-2\sqrt{(\sqrt{5}+2)(\sqrt{5}-2)}+(\sqrt{5}-2)}{\sqrt{5}-1} = \dfrac{2\sqrt{5}-2}{\sqrt{5}-1} = 2$. Therefore, the original expression, which is definitely positive, is equal to $\sqrt{2}$.

T2-Sol. 623

The prime factors of 2015 are 5, 13, and 31. Since 2015 has three prime factors, the calculation used to find the number of divisors of $2015k$ has at least three factors greater than 1. Because 18 can be expressed as such a product in one way $(3 \cdot 3 \cdot 2)$, there can be no prime factors other than $5, 13$, and 31. Two of the prime factors' exponents need to be 2 while the last prime factor's exponent is 1. Then k is equal to $5 \cdot 13$, $5 \cdot 31$, or $13 \cdot 31$. The sum of these values is 623.

T3-Sol. $44°$

O is equidistant from $\overline{AB}$ and $\overline{BC}$ because a circle centered at O is tangent to $\overline{AB}$ and $\overline{BC}$. Then O is on the angle bisector of $\angle ABC$. Then $\triangle ABO \cong \triangle CBO$ by SAS, and $\angle BCO \cong \angle BAO$. Similarly, O is equidistant from $\overline{AB}$ and $\overline{AD}$, so we also have $\angle OAB \cong OAD$. Finally, transversal $\overline{AO}$ cuts parallel lines $\overline{AB}$ and $\overline{CD}$ to form the congruent alternate interior angles $\angle BAO$ and $\angle AOD$. In $\triangle AOD$, two angles measure $68°$, so the third angle $\angle CDA$ measures $44°$.

T4-Sol. 6

Let the roots of $x^2 - bx + 10 = 0$ be r and s. From Vieta's formulas, we have $r + s = b$ and $rs = 10$. Then we have $6b = r^3 + s^3 = (r + s)(r^2 - rs + s^2) = b(b^2 - 3 \cdot 10)$. Then $6 = b^2 - 30$, and the larger solution is $b = 6$.

T5-Sol. 3

The maximum value of $\log x + \log y + \log z = \log(xyz)$ can be found by maximizing the product xyz. Using the AM-GM inequality, we have $\frac{x+4y+16z}{3} \geq \sqrt[3]{x \cdot 4y \cdot 16z}$, or $\frac{120}{3} \geq 4\sqrt[3]{xyz}$. Then $xyz \leq 1000$, so the maximum value of $\log(xyz)$ is 3.

T6-Sol. $\frac{9}{10}$

With the two allowed permutations, the third and fourth cards can never leave the third and fourth positions. However, the other three cards can be moved to any of the other three positions with an appropriate composition of these permutations. Then the desired arrangement of cards can be unshuffled from one of $3! \cdot 2! = 12$ different orders, while there are $5! = 120$ different possible arrangements. Therefore, the probability that the cards cannot be put in order is $1 - \frac{12}{120} = \frac{9}{10}$.

T7-Sol. $\frac{41}{55}$

It's easier to find the probability that the truck is unable to park, or that no unoccupied spaces are adjacent. The number of such arrangements can be found with a stars-and-bars argument, $\binom{9}{4} = 126$. There are a total of $\binom{12}{8} = 495$. The desired probability is $1 - \frac{126}{495} = \frac{14}{55}$.

T8-Sol. $3 + \sqrt{7}$

Form a monic cubic equation whose roots are x, y, and z. By Vieta's formulas, it must be $(t - x)(t - y)(t - z) = t^3 - 7t + 8t - 2$. Note that 1 is a root, and the others can be found with synthetic division and the quadratic formula. The three roots are 1, $3 + \sqrt{7}$, and $3 - \sqrt{7}$, with the largest being $3 + \sqrt{7}$.

T9-Sol. 601

Since p is a three digit number, $100 \leq p < 1000$. Then p_2 has between 7 and 10 digits and p_5 has between 3 and 5 digits (inclusive). By the condition of the ratio of digits in p_2 and p_5, p_2 must have 10 digits (so $512 \leq p < 1024$) and p_5 must have 4 digits (so $125 \leq p < 625$). Then $512 \leq p < 625$.

We also know that p_2 and p_5 have the same last two digits, and since p is prime, the last digit cannot be 0. Then we have the two cases of the last two digits being 01 and 11. Applying the Chinese Remainder Theorem, the first is satisfied by $p \equiv 1 \pmod{100}$, and the second is satisfied by $p \equiv 31 \pmod{100}$. Only 601 falls in the interval established previously. A quick check will show that 601 is prime.

T10-Sol. $90°$

The focus is $(-\frac{5}{6}, -11)$, so the line is $y = -\frac{180}{19}x - \frac{359}{19}$ and $P_2 = (-\frac{47}{57}, -\frac{4001}{361})$. Since $y' = 6x + 5$, the slopes of the tangent lines are -19 and $\frac{1}{19}$. They are negative reciprocals, so the angle formed measures $90°$.

Alternatively, think of the parabola as a reflective surface. Consider two vertical lines passing through P_1 and P_2 as light rays emanating from above. One property of parabolic curves is that rays traveling parallel to the parabola's axis of symmetry reflect from the surface and intersect at the focus. Geometrically, the incident, vertical ray and the reflected ray that passes through F form congruent angles with the tangent line at the point of tangency. We use our knowledge of vertical angles and parallel lines to show that angles $\angle P_1P_2Q$ and $\angle P_2P_1Q$ are complementary, so $\angle Q$ measures $90°$.