New York City Math Team
September 25, 2015
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Number Theory Problems

1. Find the least positive remainder (i.e. the least residue) for each of the following:

   a. $3^{12} \pmod{13}$
   b. $3^{50} \pmod{13}$
   c. $3^{2015} \pmod{13}$
   d. $16^{2015} \pmod{13}$
   e. $6^{2015} \pmod{23}$

2. Notice that 7 and 10 are relatively prime (i.e. they share no positive integer factors other than 1). Also notice that 7 had a 4-cycle (mod 10). The following steps will help you show that 4 is the largest cycle (mod 10).

   a. There are 4 numbers less than 10 that are relatively prime to 10. "1" is one of them. What are the other three? [hint: if you are stuck, look at part b :)]

   b. Prove that for any number *a* relatively prime to 10, 1*a*, 3*a*, 7*a* and 9*a* all have different residues (mod 10) and that they must be some permutation of 1, 3, 7 and 9.

   c. Show that $a^4 \equiv 1 \pmod{10}$.

   d. Find the cycles for $a = 1,3,7,9$ (mod 10).

   e. $\phi(n)$ is defined to be the number of positive integers less than *n* that are relatively prime to *n* (including 1). $\phi(10)$ is thus equal to 4. **Euler's theorem** states that $a^{\phi(n)} \equiv 1 \pmod{n}$ where *a* and *n* are relatively prime. Note that in the special case of prime *p*, $\phi(p) = p - 1$ and Euler's theorem becomes Fermat's Little Theorem. Try proving this theorem.

   f. Computing $\phi(n)$ can seem cumbersome, but there are some nice rules. Confirm that $\phi(12) = 4$. Find the cycles for each of these numbers (mod 12).

g. If $\phi(10)=4$, can you explain why $\phi(100)=40$? Hint: write down the numbers 1 through 10 and slash out the ones that share a common factor (other than 1) with 10. Write down the numbers 11 to 20 below 1 through 10. Slash out the ones with common factors. Which ones do you keep? Can you explain why this will be true for every successive set of 10 integers?

h. Thus according to Euler's theorem, what is $a^{40} \pmod{100}$ if a and 100 are relatively prime?

i. Compute the last two digits of 7^{2015}.

3. a. Show that $1^{p-1}+2^{p-1}+3^{p-1}+\cdots+(p-1)^{p-1} \equiv -1 \pmod{p}$

   b. Show that $1^{p}+2^{p}+3^{p}+\cdots+(p-1)^{p} \equiv 0 \pmod{p}$

4. Let $x=3^a+3^b$. If a and b are positive integers such that $1 \le a,b \le 100$, how many (a,b) are there such that $5|x$? [NYSML 85 T8]

5. For how many positive integer values of x, $x \le 100$, does $5|(3^x-x^2)$? [ARML 85 T5]

6. Find the remainder when $3^{\left(33^{32}\right)}$ is divided by 7. [NYSML 87 I7]

7. What is the remainder when $6^{83}+8^{83}$ is divided by 49? [Hint: what is $\phi(49)$?] [AIME 1983 #6]

8. If $15|n$, and if every digit of n is either 0 or 8, find the smallest postive integer n. [AIME 1984 #2]

9. a. Show that $7|\left(3^{105}+4^{105}\right)$.

   b. Find $3^{105}+4^{105} \pmod{11}$.

   c. Find $3^{105}+4^{105} \pmod{13}$.

10. Find all n such that $3|(2^n+1)$. [Hungary 1898 #1]

11. Prove that $1^n+2^n+3^n+4^n \equiv 0 \pmod{5}$ if n is **not** at multiple of 4. [Hungary 1901 #1]

12. If $n>1$, how many primes are greater than $n!+1$ and less than $n!+n$? Can you explain why?

13. a. Find a multiple of 3 consisting of all "1"s.

    b. Find a multiple of 7 containing all "1"s.

    c. Explain why there must be a multiple of 23 consisting of all "1"s. How many "1" are possible for this number?

    d. Explain why there must be a multiple of 41 consisting of all "3"s.

    e. If $N = 999\ldots99$ and if N is a multiple of 17, how many "9"s could N have? What is the smallest N that works? [NYCIML Senior Fall 1982 #29]