NYC MATH TEAM PRACTICE - OCTOBER 16, 2015

Solutions to Polynomials Problem Set

1.) From the remainder theorem, the remainder when $f(x)$ is divided by $(x-3)$ is just $f(3)$
$\Rightarrow f(3) = (3)^4 - 3(3)^3 + 7(3)^2 - 3 + 5 = 81 - 81 + 63 - 3 + 5 = \mathbf{65}$

2.) Let $x = 2 \Rightarrow (2)^{10} - 2(2)^5 + 3 = 1024 - 64 + 3 = \mathbf{963}$

3.) With the remainder theorem, we plug $x = -1$ into the polynomial:
$\Rightarrow (-1)^3 - k + 1 = -4 \Rightarrow k = \mathbf{4}$

4.) If $x = 15$ is a root of $P(x) = 0$, that means $P(15) = 0$, or $x - 15$ is a factor of $P(x)$
$\Rightarrow (15)^2 - 5p(15) + 6p^2 = 0$
$\Rightarrow 6p^2 - 75p + 225 = 0$
$\Rightarrow 2p^2 - 25p + 75 = 0$
$\Rightarrow (2p - 15)(p - 5) = 0$
Since $\frac{15}{2}$ is not prime, $p = \mathbf{5}$

5*a*.) Using the remainder theorem, we find $P(-3)$
$\Rightarrow P(-3) = -27 - 3k + 6 = -3k - 21$

5*b*.) If $(x + 3)$ is a factor of $P(x)$, then $x = -3$ is a root of $P(x) = 0$
Using our answer in part *a*.), $-21 - 3k = 0$
$\Rightarrow k =$ **-7**

6.) Since the remainder is 0, $p^2 + 2p(p) - 3q^2 = 0$
$\Rightarrow 3p^2 - 3q^2 = 0$
$\Rightarrow p^2 = q^2 \therefore p = \pm q$

7.) From the division algorithm, $P(x) = Q(x)(x^2 - 3x + 2) + R(x)$.
Recall that $x^2 - 3x + 2 = (x - 1)(x - 2)$

Since the divisor is a quadratic, the remainder will be a polynomial of degree 1 or 0. So we will let $R(x) = ax + b$

The sum of the coefficients is the value of $P(x)$ when $x = 1$, so $P(1) = 4$. From the remainder theorem, we have that $P(2) = 9$
$\Rightarrow P(x) = Q(x)(x - 1)(x - 2) + ax + b$
$\Rightarrow P(1) = a + b = 4$
$\Rightarrow P(2) = 2a + b = 9$

We now have a system of two equations in two variables. Solving the system, $a = 5, b = -1$
$\therefore R(x) = 5x - 1$

8.) First notice that we can factor the divisor by grouping:

$$x^2(x-2) - (x-2) = (x-2)(x^2-1) = (x-1)(x+1)(x-2)$$

With a cubic divisor, the remainder will be of degree 2,1, or 0, so let $R(x) = ax^2 + bx + c$. Using the division algorithm, we get:

$$x^{100} - 4x^{98} + 5x + 6 = Q(x)(x-1)(x+1)(x-2) + ax^2 + bx + c$$

$\Rightarrow x = 1 : a + b + c = 8$
$\Rightarrow x = -1 : a - b + c = -2$
$\Rightarrow x = 2 : 4a + 2b + c = 16$

If we subtract the first two equations, we get $2b = 10 \Rightarrow b = 5$
Now use the value of b and the 1st and 3rd equations to get the following system:

$$a + c = 3$$

$$4a + c = 6$$

Solving the system, $a = 1, c = 2 \therefore R(x) = x^2 + 5x + 2$

9.) Without finding the roots, we can use Viete's Formulas to get the sum and product of the roots.
Sum of the roots $= -\frac{9}{3} = \mathbf{-3}$
Product of the roots $= \mathbf{\frac{19}{3}}$

10.) Since one root is twice the other, call them a and $2a$. Since $k \neq 0, a \neq 0$
Using Viete's Formulas: $a + 2a = 3a = -k, (a)(2a) = 2a^2 = 2k$
$\Rightarrow k = -3a = a^2$ Since $a \neq 0$, divide by a to get that $a = \mathbf{-3}$.
The other root will be $2(-3) = \mathbf{-6}$ and k will equal $(-3)^2 = \mathbf{9}$

11.) Using Viete's Formulas on the given polynomial, we get the following:

$$a + b + c = -\frac{4}{5}, ab + ac + bc = -\frac{8}{5}, abc = -\frac{6}{5}$$

If we multiply out the given expression and rearrange terms, we get $(a+b+c)+2(ab+ac+bc)$
Sub in the correct values and you get $-\frac{4}{5} + 2(-\frac{8}{5}) = \mathbf{-4}$

12.) From Viete's Formulas, we have $r + s = 26$ and $rs = c$.
$\Rightarrow 19r + 94s = 19(r+s) + 75s = 1994$
$\Rightarrow 75s = 1994 - 19(26) = 1994 - 494 = 1500$
$\Rightarrow s = 20, r = 6 \therefore rs = \mathbf{120}$

13.) For $f(x)$: $m+n=-b$ and $mn=-9$. For $g(x)$: $-(m+n)=-a$ and $mn=c$
$\Rightarrow c=-9, a=-b$

So $f(x)=x^2+bx-9$ and $g(x)=x^2-bx-9$, which makes $h(x)=2x^2-18=0$
$\Rightarrow 2x^2=18 \Rightarrow x^2=9 \therefore x=\pm\mathbf{3}$

14.) We have $P(0)=2 \Rightarrow c=2$. That means the product of the three roots is $-c=-2$.
The sum of the roots is $-a \Rightarrow \frac{1}{3}(-a)=-2 \Rightarrow a=6$
The sum of the coefficients, $P(1)=1+6+b+2=-2 \therefore b=-\mathbf{11}$

15.) Call the roots of the polynomial $a, -a, b \Rightarrow b=3$
$\Rightarrow$ The product of the roots is $-3a^2=-\frac{d}{4}$ and $-a^2+3a-3a=-a^2=\frac{c}{4}$
$\Rightarrow \frac{-\frac{d}{4}}{\frac{c}{4}}=\frac{-3a^2}{-a^2}$
$\therefore \frac{d}{c}=-\mathbf{3}$

16.) Using Viete's Formulas, we get the following equations:

$$a+b+c=5, ab+ac+bc=7, abc=9$$

$\Rightarrow -p=2(a+b+c) \Rightarrow p=-10$
$q=(a+b)(a+c)+(a+b)(b+c)+(a+c)(b+c)$

We can multiply everything and put like terms together, but that will create some square terms and we then have to do a little extra factoring. Instead, we can rewrite the expressions as follows: $a+b=5-c$, $a+c=5-b$ and $b+c=5-a$

$$\begin{aligned}\Rightarrow q &= (5-c)(5-b)+(5-c)(5-a)+(5-b)(5-a)\\ &= 25+25+25-10(a+b+c)+(ab+bc+ac)\\ &= 75-50+7\\ &= 32\end{aligned}$$

$\therefore (p,q)=(-\mathbf{10}, \mathbf{32})$

17.) Let the roots of $P(x)=0$ be a, b, c and let the roots of $Q(x)=0$ be a, b, d

When you have the roots of a polynomial, you also have its factors and you can express the polynomial as a product of its linear factors.

$$P(x)=x^3+5x^2+px+q=(x-a)(x-b)(x-c)$$

$$Q(x)=x^3+7x^2+px+r=(x-a)(x-b)(x-d)$$

Look at the factored forms of $P(x)$ and $Q(x)$, notice that they have the factors $(x-a)$ and $(x-b)$ in common.

$$\Rightarrow Q(x)-P(x)=[(x-a)(x-b)][(x-d)-(x-c)]$$

$$= [(x-a)(x-b)](c-d)$$
$$= 2x^2 + (r-q)$$
$$\Rightarrow 2x^2 + (r-q) = (c-d)[(x-a)(x-b)]$$

$\Rightarrow$ The roots of $Q(x) - P(x) = 0$ are a and b. Because there is no x term, we conclude that $a + b = 0$

We know that $a + b + c = -5$ and $a + b + d = -7$ $\therefore (c, d) = (\mathbf{-5}, \mathbf{-7})$