Polynomials **NYC Math Team** **David Hankin**

Practice Session **October 16, 2015**

Factoring

1. Factor: a. x^4+4 b. x^4+2x^2+9 c. $x^5+x^4+x^3+x^2+x+1$.

Roots of Unity

2. Factor over $\mathcal{Z}$: a. $x^{10}+x^5+1$ b. x^5+x+1

Vieta's Formulas; Newton Sums

3. If $x+y=1$ and $xy=-3$, compute
   a. x^2+y^2 b. x^3+y^3 c. x^8+y^8
4. If a, b, and c are the roots of $x^3-x^2+2x+1=0$, find
   a. $a^2+b^2+c^2$ b. $a^3+b^3+c^3$
5. Find the sum of the 9th powers of the roots of $x^3=x+1$.

Symmetric Polynomials

6. a. Find the five 5th roots of 1 in simplest radical $a+bi$ form.
   b. Express $\cos 72°$ in simplest radical form.

Factor Theorem, Remainder Theorem

7. Let $P(x)$ be a monic polynomial with integer coefficients. If there are four different integers a, b, c, and d so that $P(a)=P(b)=P(c)=P(d)=5$, then there is no integer k so that $P(k)=8$.
8. A certain polynomial leaves a remainder of 1 when divided by $x-1$ and a remainder of 2 when divided by $x-2$. Compute the remainder when the polynomial is divided by $(x-1)(x-2)$.

Sum of Coefficients

9. Find the sum of the coefficients in the expansion of $(3x^2+x-3)^{2014}$.
10. Find the sum of the coefficients of the terms with even exponents in the expansion of $(3x^2+x-3)^{2014}$.
11. If r_1, r_2, r_3, r_4 are the zeros of $x^4+ax^3+bx^2+cx+d$, express $(r_1-1)(r_2-1)(r_3-1)(r_4-1)$ in terms of a, b, c and d.
12. If r_1, r_2, r_3, r_4 are the zeros of $x^4+ax^3+bx^2+cx+d$, express $(r_1+1)(r_2+1)(r_3+1)(r_4+1)$ in terms of a, b, c and d.
13. If r_1, r_2, r_3, r_4 are the zeros of $x^4+ax^3+bx^2+cx+d$, express $(r_1^2-1)(r_2^2-1)(r_3^2-1)(r_4^2-1)$ in terms of a, b, c and d.
14. If r_1, r_2, r_3, r_4 are the zeros of $x^4+ax^3+bx^2+cx+d$, express $(r_1^2+1)(r_2^2+1)(r_3^2+1)(r_4^2+1)$ in terms of a, b, c and d.
15. Let a, b, c, d be real numbers such that $b-d\geq 5$ and all zeros r_1, r_2, r_3, r_4 of the polynomial $P(x)=x^4+ax^3+bx^2+cx+d$ are real. Find the smallest value the product $(r_1^2+1)(r_2^2+1)(r_3^2+1)(r_4^2+1)$ can take. *[USAMO 2014]*.

PRACTICE INDIVIDUAL CONTEST

1. If $x^4 + x^3 + x^2 + x = -1$, compute x^{2015}.

2. If the ordered triple (x, y, z) that satisfies the system

$$\begin{aligned} x + y + z &= 3 \\ xy + yz + xz &= -1 \\ xyz &= -6, \end{aligned}$$

compute the maximum value of $|x - y|$.

3. For all positive integers n, Let $h(n)$ denote the sum of the nth powers of the roots of $x^2 = 3x - 1$. Compute the smallest value of $h(n)$ that is greater than 1000.

4. It is known (by me) that 25600000021 has two prime factors, the larger of which has 8 digits. Compute the larger prime factor.

5. Compute the number of ordered triples (x, y, z) of real numbers that satisfy the system:

$$\begin{aligned} x + y &= 2 \\ xy - z^2 &= 1. \end{aligned}$$

6. Let $P(x)$ be a monic cubic polynomial whose coefficients are integers. Let a, b, c and d be integers with $a < b < c$ such that $P(a) = P(b) = P(c) = 15$ and $P(d) = 22$. Compute $c - b$.

7. Express the remainder when $x^{100} - 2x^{51} + 1$ is divided by $x^2 - 1$ in simplest form.

8. Compute the sum of the imaginary roots of $x^3 + 3x^2 + 3x = 1$.

9. Solve $x^4 + 14x^2 + 1 = 7x(x^2 + 1)$.

10. Denote the 17 complex roots of $x^{17} + 12x^{12} + 4x^4 - 1 = 0$ by $r_1, r_2, \ldots, r_{17}$. Compute $(r_1^2 - 1)(r_2^2 - 1)(r_3^2 - 1) \cdots (r_{17}^2 - 1)$.

PRACTICE INDIVIDUAL CONTEST SOLUTIONS

1. The given equation implies that $x^5 = 1$, so $x^{2015} = 1$.
2. Notice that x, y, and z are roots of $w^3 - 3w^2 - w + 6 = 0$. Factor to obtain $(w-2)(w^2 - w - 3) = 0$, whose solution set is $\{2, \frac{1\pm\sqrt{13}}{2}\}$. Each of the 6 permutations of these 3 numbers corresponds to an ordered triple $x, y, z)$ that satisfies the system. Thus the requested minimum is $\frac{1+\sqrt{13}}{2} - \frac{1-\sqrt{13}}{2} = \sqrt{13}$.
3. Let a and b be the roots of the equation, and multiply both sides of the given equation by x^n to obtain $x^{n+2} = 3x^{n+1} - x^n$. Then $a^{n+2} = 3a^{n+1} - a^n$ and $b^{n+2} = 3b^{n+1} - b^n$. Add to obtain $h(n+2) = 3h(n+1) - h(n)$ for all n. Use this recursion and the fact that $h(0) = 2$ and $h(1) = 3$ to find that $h(2) = 7$, $h(3) = 18$, $h(4) = 47$, $h(5) = 123$, $h(6) = 322$, $h(7) = 843$, and $h(8) = 2207$.
4. Notice that $25600000021 = 20^8 + 20 + 1$. To factor $x^8 + x + 1$, notice that ω is a zero of the polynomial, where $\omega = \text{cis } 120°$. Thus $x^2 + x + 1$ is a factor of $x^8 + x + 1$. Then $x^8+x+1 = x^8 - x^2 + x^2 + x + 1 = x^2(x^6 - 1) + x^2 + x + 1 = x^2(x^3+1)(x^3-1) + x^2 + x + 1$ $= (x^2+x+1)[x^2(x^3+1)(x-1)+1] = (x^2+x+1)(x^6 - x^5 + x^3 - x^2 + 1)$. The requested factor is thus $20^6 - 20^5 + 20^3 - 20^2 + 1 = 60807601$.
5. Because $x+y = 2$ and $xy = z^2+1$, it follows that x and y are roots of $w^2 - 2w + (z^2+1) = 0$, whose solutions are $\dfrac{2 \pm \sqrt{-4z^2}}{2}$. Thus $z = 0$, and then $x = y = 1$, and so $(1, 1, 0)$ is the one solution to the system.
6. Notice that a, b, and c are zeros of $P(x) - 15$. Therefore $P(x) - 15 = (x-a)(x-b)(x-c)$ for all x. Then $7 = P(d) - 15 = (d-a)(d-b)(d-c)$. Because $d - a > d - b > d - c$, it follows that $d - a = 1$, $d - b = -1$, and $d - c = -7$. Then $c - b = (d-b) - (d-c) = 6$.
7. Note that the degree of the remainder polynomial is at most 1, and so let $ax + b$ denote the desired remainder. Then $x^{100} - 2x^{51} + 1 = (x^2-1)Q(x) + ax + b$. Substitute $x = 1$ to obtain $0 = a + b$, and substitute $x = -1$ to obtain $4 = -a + b$. Thus $a = -2$ and $b = 2$, and so the remainder is $-2x + 2$.
8. Add 1 to both sides to obtain $(x+1)^3 = 2$. Then $x = -1 + \sqrt[3]{2}$ is the real root of this equation, and the other two roots are imaginary. The sum of all three roots is -3, so the sum of the imaginary roots is $-3 - (-1 + \sqrt[3]{2}) = -2 - \sqrt[3]{2}$.
9. The given equation is equivalent to $\left(x^2 + \frac{1}{x^2}\right) - 7\left(x + \frac{1}{x}\right) + 14 = 0$. Let $y = x^2$. Then the equation becomes $y^2 - 2 - 7y + 14 = 0$, whose solutions are $y = 3$ and $y = 4$. Solve $x + \frac{1}{x} = 3$ and $x + \frac{1}{x} = 4$ to find the four solutions, $\frac{3\pm\sqrt{5}}{2}, 2 \pm \sqrt{3}$.
10. Note that $\displaystyle\prod_{k=1}^{17}(r_k^2 - 1) = \prod_{k=1}^{17}(r_k - 1)\prod_{k=1}^{17}(r_k + 1)$. But $P(x) = \displaystyle\prod_{k=1}^{17}(x - r_k)$, so $P(1) = \displaystyle\prod_{k=1}^{17}(1 - r_k) = -\prod_{k=1}^{17}(r_k - 1)$. Similarly, $P(-1) = \displaystyle\prod_{k=1}^{17}(-1 - r_k) = -\prod_{k=1}^{17}(r_k + 1)$. Thus $\displaystyle\prod_{k=1}^{17}(r_k^2 - 1) = P(1)P(-1) = 16 \cdot 14 = 224$.