Background

While mathematics is often used in the service of physics, occasionally physics may be used in the service of mathematics. The act of balancing simple lever can help us solve some otherwise difficult problems in geometry.

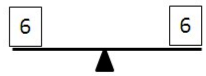

Figure 1

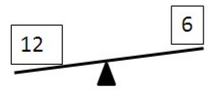

Figure 2

The lever in figure 1 is balanced because equal weights have been placed at equal distances from the fulcrum. The lever in figure 2 is not in balance because the weights on the lever are unequal while being placed at equal distances from the fulcrum. *Torque*, or *moment*, about a fulcrum is the product of the weight times the weight's distance from the fulcrum. A lever will be in balance when the torques on the lever are in balance.

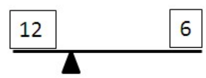

Figure 3

Note that in figure 3, moving the fulcrum closer to the heavier weight can balance the torques and therefore the lever. In this case, we simply need the ratio of the distance from the lever to the weight of 12 to the distance of the lever to the weight of 6 to be 1:2 since $12 \times 1 = 6 \times 2$.

An Application to Geometry

Consider triangle ΔABC with median $\overline{BE}$ and Cevian[1] $\overline{AD}$ dividing side $\overline{BC}$ in a ratio of 1 to 2 with the shorter segment $\overline{CD}$ as diagrammed in figure 4 below. If $\overline{AD}$ and $\overline{BE}$ meet at point P, find the ratio $BP : PE$ and $AP : PD$.

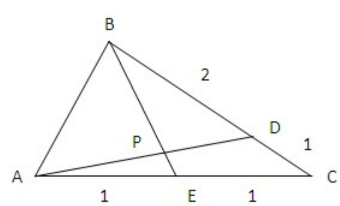

Figure 4

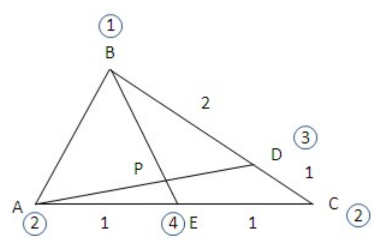

Figure 5

P, E, and D will all be balance points. First we balance torques at D for the lever $\overline{BC}$ by placing weights of 2 at B and 1 at C. This will give E an effective weight of 3. Next, balance the torques at D by placing a weight of 2 at A. The weights at A and C will give D an effective weight of 4. Figure 5 shows the triangle with the applied and effective weights. Notice that the effective weight at P, 5, is on the one hand, the sum of the weights applied to A, B and C, and on the other hand the sum of the effective weight along the Cevian $\overline{BE}$ and separately along the Cevian $\overline{AD}$.

If P is to be a balance point along the Cevian $\overline{BE}$, then the ratio of $BP : PE$ must be $4 : 1$. If P is to be a balance point along the Cevian $\overline{AD}$, then the ratio of $AP : PD$ must be the ratio $3 : 2$.

[1] A Cevian is a line segment from on vertex of a triangle to the opposite side.

1. In ΔABC, D is on $\overline{AC}$ so that $AD/DC = 5/2$, E is on $\overline{AB}$ so that $AE/EB = 2/3$, and $\overline{BD}$ and $\overline{CE}$ intersect at P. Find BP/PD and CP/PE. (DH)

2. In ΔABC, D is on $\overline{BC}$ so that $BD/DC = 5/2$, E is on $\overline{AB}$ so that $AE/EB = 4/3$, and $\overline{AD}$ and $\overline{CE}$ intersect at F. Find AF/FD and CF/FE.

3. Use mass points to prove that the medians of a triangle divide each other in a ratio of $2:1$.

4. In ΔABC, points D and E are on $\overline{BC}$ and $\overline{AC}$ respectively. If $\overline{AD}$ and $\overline{BE}$ intersect at point T so that $AT : TD = 3$ and $BT : TE = 4$, then find the ratio $CD : DB$.

5. In ΔABC, points D, E, and F are on $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ respectively. If $BD : DC = 1$, $AE : EC = 1/3$. and $AF : FB = 1/2$ and line segment $\overline{EF}$ intersects $\overline{AD}$ at P, find the ratio $AP : PD$.

Splitting Masses

The diagram in figure 6 shows ΔABC with Cevian $\overline{BF}$ and transversal $\overline{DE}$. $\overline{BF}$ divides $\overline{AC}$ in the ratio $AF : FC = 1 : 2$ as shown. $\overline{DE}$ divides $\overline{AB}$ and $\overline{BC}$ in ratios $AD : DB = 2 : 5$ and $BE : EC = 3 : 2$. The Cevian and the transversal intersect at point P. By splitting the mass at the common vertex of the transversals and balancing $\overline{AB}$ separately from $\overline{BC}$, we will be able to find the ratios $BP : PF$ and $DP : PE$.

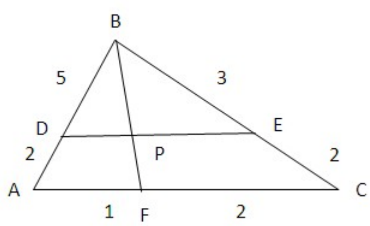

Figure 6

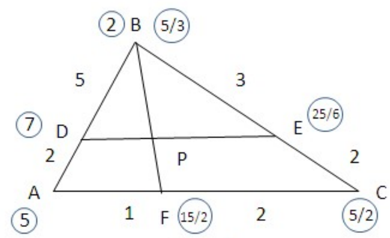

Figure 7

Starting at B, assign a weight of 2 to B and a weight of 5 to A to balance the torques at D. D will then have an effective weight of 7. Next balance the torques at F by assigning the weight $\frac{5}{2}$ to C. This will give an effective weight of $\frac{15}{2}$ at F. The last step is to balance the torques at E by adding mass of $\frac{5}{3}$ to point B. This will give a total mass of $\frac{11}{3}$ at B and an effective mass of $\frac{25}{6}$ at E. See figure 7 above.

Note that the total effective weight at P is the sum of the weights at A, B, and C, and is equal to $\frac{67}{6}$. This is the sum of the effective weights along the segment $\overline{BF}$ and separately along the segment $\overline{DE}$. Finally the ratio $BP : PF = \frac{15}{2} : \frac{11}{3}$ or $45 : 22$ and $DP : PE = \frac{25}{6} : 7$ or $25 : 42$

6. In ΔABC point E is on $\overline{AB}$ such that $AE : EB = 4 : 3$. Point D is on $\overline{BC}$ such that $BD : DC = 5 : 2$. Point G is on $\overline{AC}$ with $AG : GC = 3 : 7$. Cevian $\overline{BG}$ and transversal $\overline{ED}$ intersect at point P. Find the ratios $BP : PG$ and $EP : PD$.

7. In ΔABC, M is on $\overline{AC}$, $AM/MC = 1/2$, N is on $\overline{AB}$, $AN/NB = 2$, P is on $\overline{MN}$, and $MP/PN = 3/2$. Assign masses to A, B, and C so that P is their center of mass. (DH)