NYC Math Team, Inequality Techniques, lecture by Jan Siwanowicz

Hölder's Inequality for an Array

Given an array of nonnegative numbers, the geometric mean of the arithmetic means of the numbers in each column is greater than or equal to the arithmetic mean of the geometric means of the numbers in each row.

H1. Show that if x_1, x_2, … , x_n are nonnegative, then

a. $\left(x_1^n+(n-1)\right)\left(x_2^n+(n-1)\right)\ldots\left(x_n^n+(n-1)\right)\geq\left(x_1+x_2+\ldots+x_n\right)^n$

b. $$\left(x_1+x_2^3+x_3^5+\ldots+x_n^{2n-1}\right)\left(x_1^3+x_2^5+x_3^7+\ldots+x_{n-1}^{2n-1}+x_n\right)\left(x_1^5+x_2^7+x_3^9+\ldots+x_{n-2}^{2n-1}+x_{n-1}+x_n^3\right)\times$$
$$\times\ldots\times\left(x_1^{2n-1}+x_2+x_3^3+\ldots+x_n^{2n-3}\right)\geq\left(x_1^n+x_2^n+\ldots+x_n^n\right)^n$$

c. $(1+x_1)(1+x_2)\ldots(1+x_n)\geq\left(1+\sqrt[n]{x_1x_2\ldots x_n}\right)^n$

H2. Positive numbers x, y, z satisfy $x+y+z=1$. Show that $\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right)\geq 64$.

H3. Positive numbers x, y, z satisfy $xyz=1$. Show that $(x+2y)(y+2z)(z+2x)\geq 27$.

H4. Positive numbers x_1, x_2, … , x_n satisfy $x_1+x_2+\ldots+x_n=1$. Show that

a. $\left(1+\frac{1}{x_1}\right)\left(1+\frac{1}{x_2}\right)\ldots\left(1+\frac{1}{x_n}\right)\geq(n+1)^n$

b. $\left(x_1+\frac{1}{x_1}\right)^2+\left(x_2+\frac{1}{x_2}\right)^2+\ldots+\left(x_n+\frac{1}{x_n}\right)^2\geq\frac{\left(n^2+1\right)^2}{n}$

c. $\left(x_1+\frac{1}{x_1}\right)^n+\left(x_2+\frac{1}{x_2}\right)^n+\ldots+\left(x_n+\frac{1}{x_n}\right)^n\geq\frac{\left(n^2+1\right)^n}{n^{n-1}}$

d. $\left(\frac{1}{x_1^2}-1\right)\left(\frac{1}{x_2^2}-1\right)\ldots\left(\frac{1}{x_n^2}-1\right)\geq\left(n^2-1\right)^n$

H5. Show that for any positive numbers a, b, c we have

$$3+(a+b+c)+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq\frac{3(a+1)(b+1)(c+1)}{abc+1}$$

Rearrangement Inequalities

The sum of the products of corresponding terms of sequences is the greatest when the terms in the sequences are ordered identically. For two sequences, the sum of the products of the corresponding terms is least when the terms in the sequences are arranged in opposite order.

R1. Show that if a and b are positive, $a^3+b^3\geq a^2b+ab^2$

R2. Show that if a and b are nonzero, $a^4+b^4\leq\frac{a^6}{b^2}+\frac{b^6}{a^2}$

R3. Show that if a and b are positive, $\frac{1}{a^3}+\frac{1}{b^3}\le\frac{1}{a^3}\sqrt{\frac{b}{a}}+\frac{1}{b^3}\sqrt{\frac{a}{b}}$

R4. Show that if a and b are positive, $\frac{a^3}{b}+\frac{b^3}{a}\ge a^2+b^2$

R5. Show that if a and b are positive, $\sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}}\ge\sqrt{a}+\sqrt{b}$

R6. Show that if a and b are positive, $\frac{a}{b}+\frac{b}{a}\ge 2$

R7. Show that if a and b are positive, n natural, $a^n+b^n\ge a^{n-1}b+ab^{n-1}$

R8. Show that if a, b, c, and d are positive, $a^{c+d}+b^{c+d}\ge a^cb^d+a^db^c$

R9. Show that if a, b, and c are positive, $a^3+b^3+c^3\ge a^2b+b^2c+c^2a$

R10. Show that if a, b, and c are positive, $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{3}{2}$

R11. Show that if a, b, and c are positive, $\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\ge\frac{a+b+c}{2}$

R12. Show that if a, b, and c are positive,

a. $2\left(a^3+b^3+c^3\right)\ge ab\left(a+b\right)+bc\left(b+c\right)+ca\left(c+a\right)$

b. $a+b+c\le\frac{a^2+b^2}{2c}+\frac{b^2+c^2}{2a}+\frac{c^2+a^2}{2b}\le\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}$

R13. Show that if a, b, and c are positive, $\frac{a^8+b^8+c^8}{a^3b^3c^3}\ge\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

R14. Show that if a, b, and c are positive, $a^3b+b^3c+c^3a\ge a^2bc+b^2ca+c^2ab$

R15. Show that if a, b, and c are positive, $\frac{a^3b}{c}+\frac{a^3c}{b}+\frac{b^3a}{c}+\frac{b^3c}{a}+\frac{c^3a}{b}+\frac{c^3b}{a}\ge 6abc$

R16. Show that if a, b, and c are positive, $\frac{a}{b^4}+\frac{b}{c^4}+\frac{c}{a^4}\ge\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}$

R17. Show that if a, b, and c are positive, $\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\ge\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}$

R18. Show that if a, b, and c are positive, $ab\left(a+b\right)+bc\left(b+c\right)+ca\left(c+a\right)\ge 6abc$

R19. Show that if a, b, c, and d are positive, $a^3+b^3+c^3+d^3\ge a^2b+b^2c+c^2d+d^2a$

R20. $n\ge 3$, positive numbers x_1, x_2, ... , x_n satisfy $x_1+x_2+...+x_n=s$, show that

$\frac{x_1}{s-x_1}+\frac{x_2}{s-x_2}+...+\frac{x_n}{s-x_n}\ge\frac{n}{n-1}$

R21. Show that if x_1, x_2, ... , x_n are positive, $\frac{x_1}{x_2}+\frac{x_2}{x_3}+...+\frac{x_n}{x_1}\ge n$

R22. (Chebychev's Inequality) Show that if $a_1 \ge a_2 \ge ... \ge a_n$ and $b_1 \ge b_2 \ge ... \ge b_n$ then
$n(a_1b_1 + a_2b_2 + ... + a_nb_n) \ge (a_1 + a_2 + ... + a_n)(b_1 + b_2 + ... + b_n)$

R23. Show that if x_1, x_2, … , x_n are positive, $n(x_1^2 + x_2^2 + ... + x_n^2) \ge (x_1 + x_2 + ... + x_n)^2$

R24. Show that if a, b, and c are positive, $\frac{a^3+b^3+c^3}{a^2+b^2+c^2} \ge \frac{a+b+c}{3}$

R25. Show that if x_1, x_2, … , x_n are positive, $(x_1 + x_2 + ... + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) \ge n^2$

R26. Show that if x_1, x_2, … , x_n are positive, $\frac{x_1 + x_2 + ... + x_n}{n} \ge \left(\frac{x_1^{-2} + x_2^{-2} + ... + x_n^{-2}}{n}\right)^{\frac{-1}{2}}$

R27. Positive numbers x_1, x_2, … , x_n satisfy $x_1 + x_2 + ... + x_n = 1$. Show that
$\frac{x_1}{\sqrt{1-x_1}} + \frac{x_2}{\sqrt{1-x_2}} + ... + \frac{x_n}{\sqrt{1-x_n}} \ge \sqrt{\frac{n}{n-1}}$

R28. Show that if a, b, and c are positive, $9(a^3 + b^3 + c^3) \ge (a+b+c)^3$

R29. Show that if x_1, x_2, … , x_n are positive,

a. $x_1^n + x_2^n + ... + x_n^n \ge x_1^{n-1}x_2 + x_2^{n-1}x_3 + ... + x_n^{n-1}x_1$

b. $\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + ... + \frac{x_n^2}{x_1} \ge x_1 + x_2 + ... + x_n$

R30. Show that if a, b, and c are positive, $a^7 + b^7 + c^7 \ge a^2b^2c^2(a+b+c)$

R31. (AM-GM) Inequality) Show that if x_1, x_2, … , x_n are nonnegative, $\frac{x_1 + x_2 + ... + x_n}{n} \ge \sqrt[n]{x_1x_2...x_n}$

R32. Generalize R22 to the case of multiple sequences.

R33. Show that if x_1, x_2, … , x_n are positive, and m is natural,
$n^{m-1}(a_1^m + a_2^m + ... + a_n^m) \ge (a_1 + a_2 + ... + a_n)^m$

R34. Show that if a, b, and c are positive, $3(a^3 + b^3 + c^3) \ge (a+b+c)(a^2 + b^2 + c^2)$

R35. Show that if a_1, a_2, … , a_n are positive,

a. $\frac{a_1}{a_2^2} + \frac{a_2}{a_3^2} + ... + \frac{a_n}{a_1^2} \ge \frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}$

b. $n(a_1^{k+m} + a_2^{k+m} + ... + a_n^{k+m}) \ge (a_1^k + a_2^k + ... + a_n^k)(a_1^m + a_2^m + ... + a_n^m)$ {k and m natural}

c. $(n-1)(a_1^m + a_2^m + ... + a_n^m) \le \frac{a_2^k + a_3^k + ... + a_n^k}{a_1^{k-m}} + \frac{a_1^k + a_3^k + ... + a_n^k}{a_2^{k-m}} + ... + \frac{a_1^k + a_2^k + ... + a_{n-1}^k}{a_n^{k-m}}$

R36. Show that if a, b, and c are positive, $a+b+c \le \frac{a^4+b^4+c^4}{abc}$

R37. Show that if a, b, and c are nonnegative,
$a^3+b^3+c^3+3abc \ge a^2(b+c)+b^2(c+a)+c^2(a+b)$

R38. Show that if a, b, and c are nonnegative, and $a+b+c=1$, $\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1} \le \frac{3}{4}$

R39. Show that if a, b, and c are positive, $\sqrt{\frac{a+b}{c}}+\sqrt{\frac{b+c}{a}}+\sqrt{\frac{c+a}{b}} \ge 3\sqrt{2}$

R40. Show that if a, b, and c are positive, $\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b} \ge 4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$

Triangle side substitution

If a, b, and c are the sides of a triangle, there exist unique positive x, y, and z, such that $a=x+y$, $b=y+z$, and $c=z+x$. The converse is also true.

Let a, b, and c be the sides of a triangle with area K, semiperimeter s, and inradius r.

T1. Show $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le \frac{1}{a+b-c}+\frac{1}{b+c-a}+\frac{1}{c+a-b}$

T2. Show $\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b} \le \sqrt{a}+\sqrt{b}+\sqrt{c}$

T3. Show $a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c) \le 3abc$

T4. Show $a^2b(a-b)+b^2c(b-c)+c^2a(c-a) \ge 0$

T5. Show $a^2+b^2+c^2 \ge 4K\sqrt{3}$

T6. Show $2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \ge \frac{a}{c}+\frac{c}{b}+\frac{b}{a}+3$

T7. Show $\frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c} \ge 3$

T8. Show $a^2(2b+2c-a)+b^2(2c+2a-b)+c^2(2a+2b-c) \ge 9abc$

T9. $a(b-c)^2+b(c-a)^2+c(a-b)^2+4abc > a^3+b^3+c^3$

T10. x, y, and z positive, such that $xyz(x+y+z)=1$. Determine minimum value of $(x+y)(x+z)$

T11. Show

a. $2(ab+bc+ca) > a^2+b^2+c^2$

b. $ab+bc+ca \ge 4K\sqrt{3}$

c. $(s-a)^{-2}+(s-b)^{-2}+(s-c)^{-2} \ge r^{-2}$

d. $3r^2\sqrt{3} \le K \le \frac{s^2}{3\sqrt{3}}$