NEW YORK CITY MATH TEAM

Problem Sets

1. Prove that for $k \geq 6$, we have $3^k > 2^{k+3}$.
2. Prove that $\sqrt{2}$ is irrational.
3. Let n be a positive integer. Prove that $1 + 8 + 27 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.
4. Prove that the area of a triangle is equal to its inradius times its semiperimeter.
5. Prove *Bernoulli's Inequality*, which states that if $x \geq -1$ then $(1+x)^n \geq 1 + nx$ for all natural numbers n. (Hint: induction)
6. Let $ABCD$ be a quadrilateral with an inscribed circle. Prove that $AB + CD = AD + BC$.
7. The numbers $1, 2, \ldots, 2010, 2011$ are written on the board. It is permitted to take two numbers a, b and replace them with $a - b$. Prove that if this operation is applied until there is only one number on the board, then this number is even.
8. Prove that among any six people, there are always three who know each other or three who are complete strangers.
9. Two rows of ten pegs are lined up and adjacent pegs are spaced 1 unit apart. Determine, with proof, the number of ways in which ten rubber-bands be looped around the pegs so that no peg does not contain a rubber band. (Rubber bands cannot stretch more than $\sqrt{2}$ units.)
10. Let F_n be the sequence defined by $F_0 = 0, F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Prove that $F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$ for all nonnegative integers m, n.