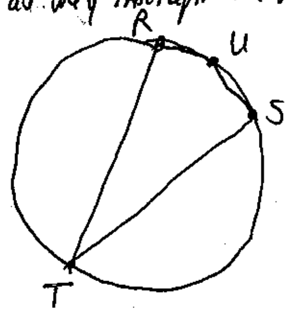
① This is not necessarily true. Yes, both angles are inscribed angles, and thus should have the same measure as they intercept the same arc, but we can engineer a situation where this is not true:

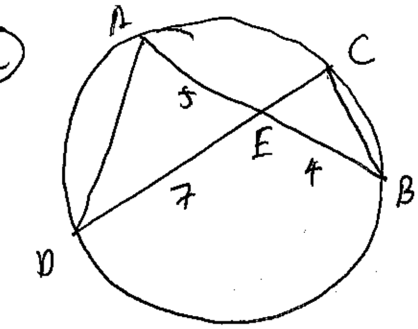


Note T is "outside" of the minor arc $\overset{\frown}{RS}$, but U is between R and S. Thus

$$m\angle RTS = \tfrac{1}{2} m\overset{\frown}{RS} = \tfrac{1}{2}\cdot 50 = 25^\circ, \text{ but}$$
$$m\angle RUS = \tfrac{1}{2} m\overset{\frown}{RTS} = \tfrac{1}{2}\cdot(360-50) = \tfrac{1}{2}\cdot 310 = 155^\circ!!$$

So, before invoking the fact that two inscribed angles intercepting the same arc are congruent, make sure that they are both, indeed, the same arc!!

②



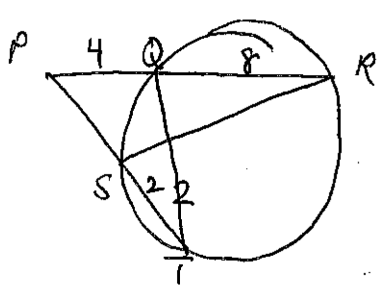
Draw in chords $\overline{AD}$ and $\overline{BC}$ (the others work too.) Note that $\triangle AED \sim \triangle CEB$ (you can use the vertical $\angle AED$ and $\angle CEB$, and the fact that $\angle A$ and $\angle C$ both intercept $\overset{\frown}{DB}$, so they are $\cong$) by AA, so we can write proportions:

$$\frac{AE}{EC} = \frac{DE}{EB} \longrightarrow \frac{5}{EC} = \frac{7}{4} \longrightarrow 7EC = 20$$

(or $AE\cdot EB = DE\cdot EC$) $\qquad EC = \frac{20}{7}$.

So, $DC = DE + EC = 7 + \frac{20}{7} = \frac{49+20}{7} = \boxed{\frac{69}{7}.}$

③



Taking inspiration from last time, we draw chords $\overline{QT}$ and $\overline{RS}$. Note that since the triangles share $\angle P$, and both $\angle T$ and $\angle R$ intercept the same arc $\overset{\frown}{SQ}$, $\triangle PQT \sim \triangle PSR$ by AA.

So, proportions again:

$$\frac{PS}{PQ} = \frac{PR}{PT} \longrightarrow \frac{x}{4} = \frac{12}{x+2} \longrightarrow x(x+2) = 48$$

$x^2 + 2x = 48$

$x^2 + 2x - 48 = 0$

$(x+8)(x-6) = 0$

$x = 6, -8$ (reject, as x is a length.)

(or $PS\cdot PT = PQ\cdot PR$)

So, $\boxed{PS = 6.}$

④ As T approaches S, PT becomes closer in length to PS, while PS remains stable. Eventually, once T and S occupy the same point, PT and PS have the same length, so we get that

$$\frac{PS}{PQ} = \frac{PR}{PS}, \text{ or } PS^2 = PQ\cdot PR.$$

This leads us to a general statement of Power of a Point:

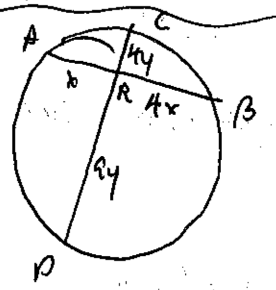
** The Generalized Power of a Point Theorem:

Let P be a point in the same plane as a circle O, and let a line passing through P intersect circle O at two points, W and X. Then, the product

$$PW \cdot PX$$

is a constant for any line that passes through P. Note this is also true if W and X are the same points!

5



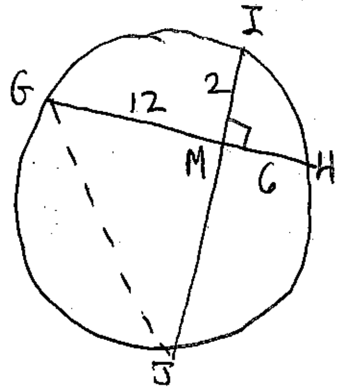
By Power of a Point,

$$AR \cdot RB = CR \cdot RD$$
$$x \cdot 4x = 4y \cdot 9y$$
$$4x^2 = 36y^2$$
$$x^2 = 9y^2$$
$$x = 3y \text{ (reject negative)}$$

We seek $\frac{AB}{CD}$. This is

$$\frac{AB}{CD} = \frac{5x}{13y} = \frac{5 \cdot 3y}{13y} = \frac{5 \cdot 3}{13} = \boxed{\frac{15}{13}.}$$

6



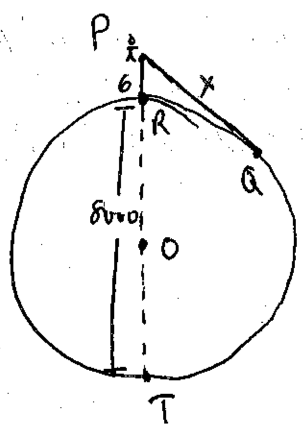
Note that $HM = 6$ and $IM = 2$, so by Power of a Point,

$$GM \cdot MH = IM \cdot MJ$$
$$12 \cdot 6 = 2 \cdot MI$$
$$72 = 2 \cdot MI$$
$$MI = 36.$$

As $\overline{GH} \perp \overline{IJ}$, M is a right angle, so:

$$GJ^2 = GM^2 + MJ^2$$
$$GJ^2 = 12^2 + 36^2$$
$$= 12^2 + 3^2 12^2$$
$$GJ^2 = 10 \cdot 12^2$$
$$GJ = \sqrt{10 \cdot 12^2} = \sqrt{12^2} \cdot \sqrt{10} = \boxed{12\sqrt{10}.}$$

7

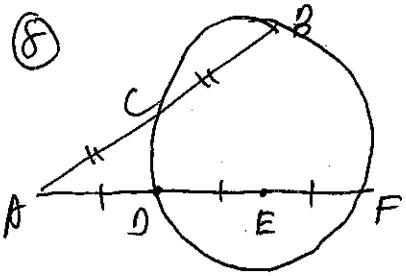


Let's use the theorem, where x is the distance to Q. So:

$$PQ^2 = PR \cdot PT$$
$$x^2 = 6(8006)$$
$$x^2 = 48{,}048$$
$$x \approx \sqrt{48{,}000} \approx \sqrt{1600} \cdot \sqrt{30} = 40\sqrt{30} \text{ mi}$$
$$\approx 219.09 \text{ mi} \approx \boxed{219 \text{ mi}.}$$

Note that 6 is very small compared to the diameter of the earth, so omitting the 48 from the end of the 48,000 will result in a rather insignificant error, so we can safely do this. (It asked for an approximation, after all)

8

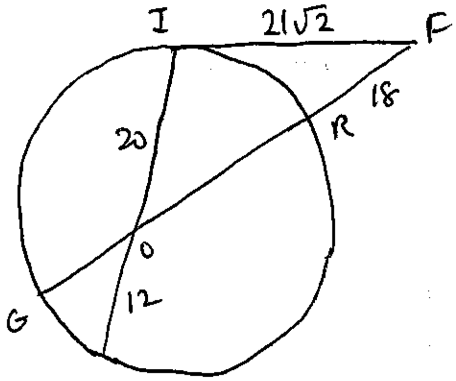


Let $AC = x$ and $AD = y$. By Power of a Point,

$$AC \cdot AB = AD \cdot AF$$
$$x \cdot 2x = y \cdot 3y$$
$$2x^2 = 3y^2$$
$$\frac{2x^2}{y^2} = 3$$

$$\frac{x^2}{y^2} = \frac{3}{2} \longrightarrow \frac{x}{y} = \sqrt{\frac{3}{2}}$$
$$= \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \boxed{\frac{\sqrt{6}}{2}.}$$

9



We look at Power of a Point at F first:

$$FI^2 = FR \cdot FG$$
$$(21\sqrt{2})^2 = 18 \cdot FG$$
$$FG = \frac{21^2 \cdot \cancel{2}}{\cancel{18}_9} = \frac{21^2}{3^2} = \left(\frac{21}{3}\right)^2 = 7^2 = 49.$$

Thus, $RG = FG - FR = 49 - 18 = 31.$

Now, let $OR = x$ and $GO = 31 - x$. So, taking the Power around O, we have

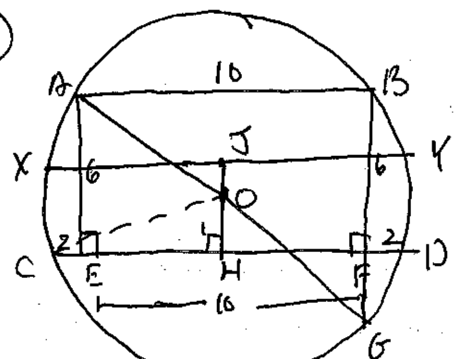
$$OR \cdot OG = OI \cdot ON$$
$$x(31 - x) = 20 \cdot 12$$
$$31x - x^2 = 240$$

$$x^2 - 31x + 240 = 0$$
$$(x-16)(x-15) = 0$$
$$x = 15 \text{ or } 16.$$

As $OR > GO$,

$$\boxed{OR = 16.}$$

10



Draw the chords. Let E and F be on $\overline{CD}$ such that $\overline{AE} \perp \overline{CD}$ and $\overline{BF} \perp \overline{CD}$. Note that by symmetry, $FD = CE = \frac{1}{2}(14 - 10) = 2$. Now, let's extend $\overline{BF}$ until it intersects the circle again at G.

By Power of a Point around F,

($\overline{XY}$ is our target chord.)

$$BF \cdot FG = FD \cdot CF$$
$$6 \cdot FG = 2 \cdot 12.$$
$$FG = \frac{2 \cdot 12}{6} = 4.$$

Note here that $FG = 4$, so $BG = 6 + 4 = 10$. Since $m\angle ABG = 90°$, and it is an inscribed angle, by Thales' Theorem $\widehat{ACG}$ is a semicircle, making $\overline{AG}$ a diameter. $AG = 10\sqrt{2}$ as $\triangle ABG$ is isosceles, so the circle has radius $5\sqrt{2}$. Let H be the foot of a perpendicular segment $\overline{OH}$ to chord $\overline{CD}$. H must be the midpoint of CD, so $CH = 7$. Examining right $\triangle OHC$, we have

$$OC^2 = OH^2 + HC^2$$
$$(5\sqrt{2})^2 = OH^2 + 7^2$$
$$50 = OH^2 + 49$$
$$OH = 1.$$

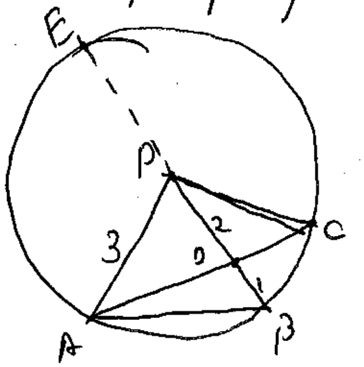
Thus, $\overline{XY}$, the target, lies 2 units above the center. Let J be the foot of a perp. segment $\overline{OJ}$ to $\overline{XY}$. So with right $\triangle OXJ$,

$$OX^2 = OJ^2 + XJ^2 \longrightarrow (5\sqrt{2})^2 = 2^2 + XJ^2$$

$$XJ^2 = 46$$
$$XJ = \sqrt{46}.$$

$$\boxed{\text{So } XY = 2\sqrt{46}.}$$

(11) What a strange goal. The goal being a product suggests Power of a Point at D, but there exists no circle! We can construct a circle centered at P passing through A — this is supported by the fact that $PA = PB$, $\angle APB$ would be a central angle, and $\angle ACB$ would be inscribed, making it one-half of the central angle that also intercepts the arc. Thus, let us draw a circle — centered at P, and passing through A, B, and C:



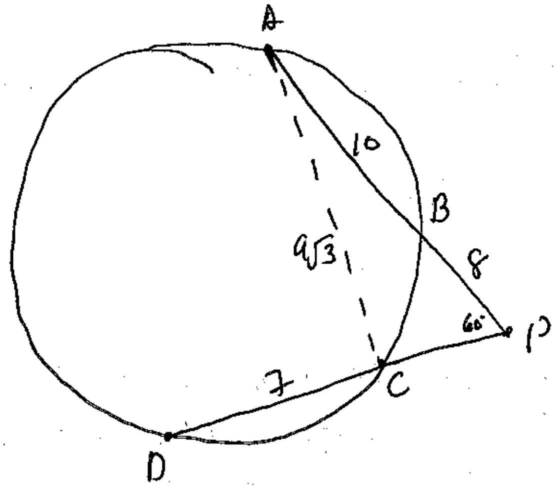
We would love to use Power of a Point at D, but sadly DP is an incomplete piece of incomplete chord $\overline{PB}$. Thus, let us finish it, by extending $\overline{BP}$ through P to hit the circle at E.

The circle has radius 3, so $PE = 3$, and $PD = 3+2 = 5$.

So, by Power of a Point at D,

$$AD \times CD = BD \times ED$$
$$= 1 \times 5 = \boxed{5.}$$

(12)



Well, the given information suggests Power at P, so let's do that:

$$PC \cdot PD = PB \cdot PA$$
$$PC \cdot (PC+7) = 8 \cdot 18$$
$$PC^2 + 7PC = 144$$
$$PC^2 + 7PC - 144 = 0$$
$$(PC+16)(PC-9) = 0$$
$$PC = -16 \text{ or } PC = 9.$$

We discard the negative value, so $PC = 9$.

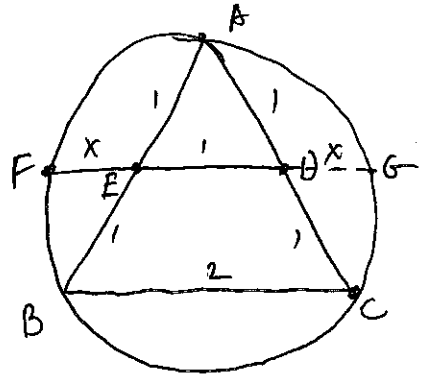
Now, in $\triangle PCA$, we have $PC = 9$, and $PA = 10+8 = 18$, with a $60°$ angle between them. This is a hallmark of a 30-60-90 triangle, as the longest side is double the shortest side with the $60°$ angle sandwiched in between. Thus, $\overline{AC} \perp \overline{DP}$, and $AC = 9\sqrt{3}$. Since $\angle ACD$ is a right angle, again by Thales' Theorem, $\overset{\frown}{AD}$ must be a semicircle, as $\angle ACD$ is an inscribed angle that intercepts it. Thus, $\overline{AD}$ is a diameter. So, by Pythagorean Theorem,

$$AD^2 = AC^2 + CD^2$$
$$= (9\sqrt{3})^2 + 7^2 = 81 \cdot 3 + 49 = 243 + 49 = 292.$$

$AD = \sqrt{292} = \sqrt{4} \cdot \sqrt{73} = 2\sqrt{73}$. The radius is thus $\sqrt{73}$, so the circle's area is

$$A = \pi r^2 = \pi(\sqrt{73})^2 = \boxed{73\pi.}$$

(13)



Without loss of generality, let the triangle have side of length 2. Again, we'd love to use Power of a Point, preferably at E, but the chords are again incomplete. So, let's extend $\overline{DE}$ the other way to strike the circle again at G. Let $EF = x$. Thus, by symmetry, $DG = x$ as well. Note that as $\triangle ABC$ is equilateral, $AE = ED = AD = 1$, so we can solve for this. Using Power at E, we have:

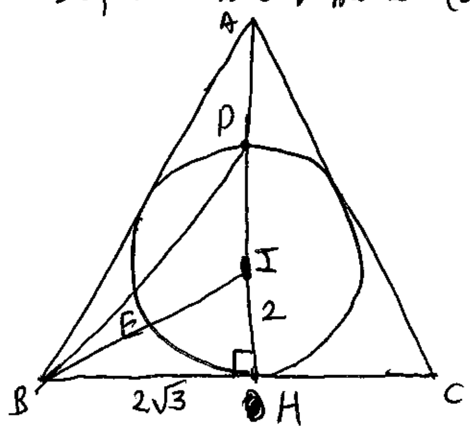
$$AE \cdot BE = FE \cdot EG$$
$$1 \cdot 1 = x(1+x)$$
$$1 = x + x^2 \longrightarrow x^2 + x - 1 = 0$$
$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

We discard the negative root, so $x = EF = \dfrac{-1+\sqrt{5}}{2}.$

The desired ratio is thus:
$$\frac{DE}{EF} = \frac{1}{\frac{-1+\sqrt{5}}{2}} = \frac{2}{-1+\sqrt{5}} \cdot \frac{-1-\sqrt{5}}{-1-\sqrt{5}} = \frac{-2-2\sqrt{5}}{1-5} = \frac{-2-2\sqrt{5}}{-4} = \frac{1+\sqrt{5}}{2}.$$

So, the ordered triple is $(a, b, c) = \boxed{(1, 5, 2)}.$

(14)



Let H be the foot of the altitude from A to $\overline{BC}$. Let I be the center of the incircle. Since $\triangle ABC$ is equilateral, it is not hard to show that altitude $\overline{AH}$ is also a median and an angle bisector. The same is true for segment $\overline{BI}$, when drawn. Thus, $m\angle IBH$ is $30°$, and we have a 30-60-90 $\triangle$ in $\triangle IBH$. Since $IH = 2$, we can see that $BH = IH\sqrt{3} \longrightarrow 2\sqrt{3}$, so the triangle has side $4\sqrt{3}$. Also note that $\triangle DHB$ is right, so

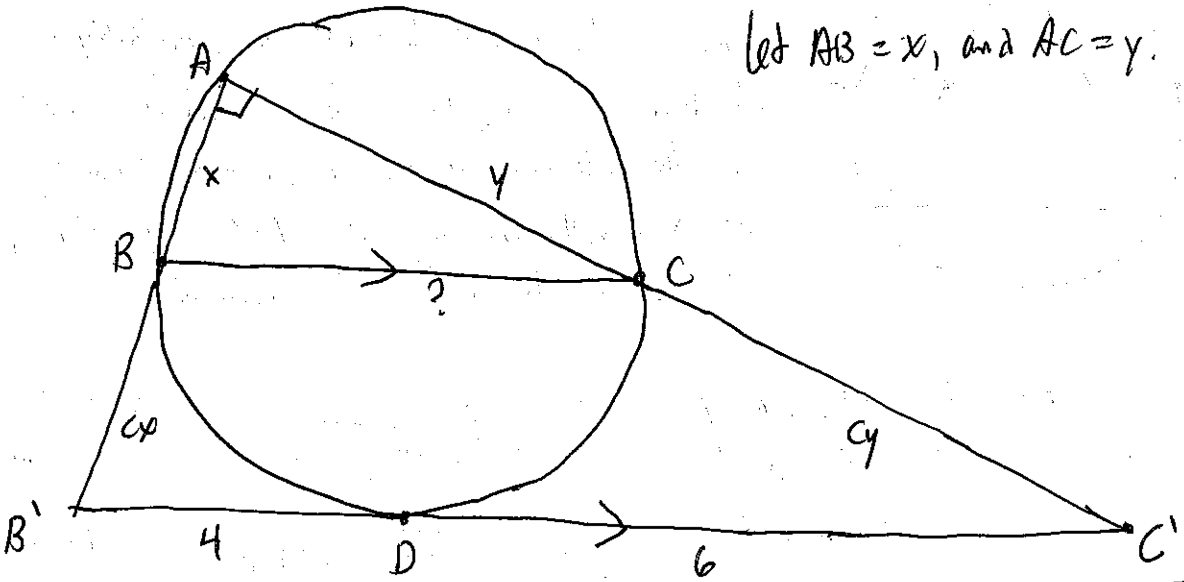
$$DB^2 = BH^2 + DH^2$$
$$DB^2 = (2\sqrt{3})^2 + 4^2$$
$$= 12 + 16 = 28.$$

Thus, $DB = \sqrt{28} = \sqrt{4} \cdot \sqrt{7} = 2\sqrt{7}.$

So, by Power of a Point on B, we have:

$$BH^2 = BE \cdot BD$$
$$(2\sqrt{3})^2 = BE \cdot 2\sqrt{7}$$
$$12 = BE \cdot 2\sqrt{7} \longrightarrow BE = \frac{12}{2\sqrt{7}} = \frac{6}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \boxed{\frac{6\sqrt{7}}{7}}.$$

(15) We start with a careful sketch:



let $AB = x$, and $AC = y$.

$\overline{BC}$ is a diameter, so $\triangle ABC$ is right. Thus $BC = \sqrt{x^2+y^2}$. Since $\overline{BC} \parallel \overline{B'C'}$, we know that $\overline{BC}$ divides the sides of $\triangle AB'C'$ proportionally, so let's let the constant of proportionality be c. So $BB' = cx$ and $CC' = cy$. The two triangles are similar, so $\triangle ABC \sim \triangle AB'C'$ with a scale factor of $(1+c):1$.

Since $B'C' = 10$, we know that $\frac{B'C'}{BC} = 1+c$, so we have $\cancel{\sqrt{x^2+y^2}}$ $\frac{10}{\sqrt{x^2+y^2}} = 1+c$, and

$$\cancel{\text{Bec}} \quad 10 = (1+c)\sqrt{x^2+y^2}.$$

We now turn our attention to B' and C'. Noting $B'D$ is a tangent and $B'A$ is a secant, we have:

$$B'D^2 = B'B \cdot B'A$$
$$16 = cx(cx+x)$$
$$16 = cx^2(c+1)$$
$$16 = c(c+1)(x^2).$$

Similar logic on C' yields:

$$C'D^2 = C'C \cdot CA$$
$$36 = cy(cy+y)$$
$$36 = c(c+1)y^2.$$

Noting that our goal contains a $\sqrt{x^2+y^2}$, we are motivated to add the ~~two~~ equations:

$$\begin{array}{r} 16 = c(c+1)x^2 \\ +\ 36 = c(c+1)y^2 \\ \hline 52 = c(c+1)(x^2+y^2). \end{array}$$

From before, we have $\sqrt{x^2+y^2} = \frac{10}{1+c}$, so $x^2+y^2 = \frac{100}{(1+c)^2}$.

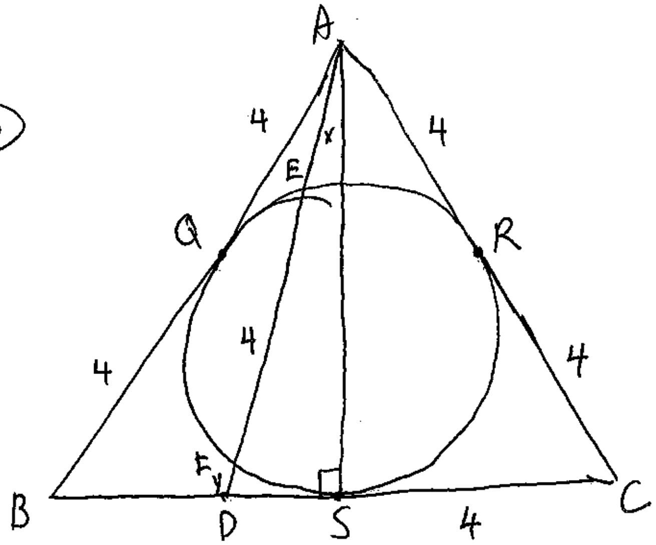
Substitute:

$$52 = c\cancel{(c+1)}\frac{100}{(1+c)^{\cancel{2}}}$$
$$52 = \frac{100c}{1+c}$$
$$52 + 52c = 100c$$
$$52 = 48c$$
$$c = \frac{52}{48} = \frac{13}{12}.$$

Thus, our goal is

$$BC = \sqrt{x^2+y^2} = \frac{10}{1+c}$$
$$= \frac{10}{1+\frac{13}{12}} = \frac{10}{\frac{25}{12}}$$
$$= \frac{\cancel{10}^{2}\cdot 12}{\cancel{25}_{5}} = \boxed{\frac{24}{5}.}$$

~~13~~ 16



let $x = AE$ and $y = FD$.

We can definitely solve for x by looking at A.

By computing Power of a Point at A, we have:

$$AQ^2 = AE \cdot AF$$
$$16 = x(x+4)$$
$$16 = x^2+4x$$
$$0 = x^2+4x-16$$
$$x = \frac{-4 \pm \sqrt{4^2-4(1)(-16)}}{2(1)}$$
$$= \frac{-4\pm\sqrt{16+64}}{2} = \frac{-4\pm\sqrt{80}}{2} = \frac{-4\pm4\sqrt{5}}{2}.$$
$$= -2\pm2\sqrt{5}.$$

We discard the negative root, so $x = AE = 2\sqrt{5}-2$.

To determine y, we'd have to use right $\triangle ADS$, (Note $\overline{AS}$ is an altitude), but we lack DS. However, noting D, we can use Power there:

$$DS^2 = DF \cdot DE \rightarrow DS^2 = y(y+4).$$

So, we get:

$$DS^2+AS^2 = AD^2.$$
$$y(y+4)+(4\sqrt{3})^2 = (x+4+y)^2$$
$$\cancel{y^2}+4y+48 = x^2+2xy+8x+4y+\cancel{y^2}+16$$
$$32 = x^2+2xy+8x+4y$$
$$32 = (2\sqrt{5}-2)^2+2(2\sqrt{5}-2)y+8(2\sqrt{5}-2)+4y$$
$$32 = 20-8\sqrt{5}+4+(4\sqrt{5}-4)y+16\sqrt{5}-16+4y$$
$$32 = 8+8\sqrt{5}+4\sqrt{5}y\cancel{+4y-4y}$$
$$y = \frac{24-8\sqrt{5}}{4\sqrt{5}} = \frac{24}{4\sqrt{5}}-\frac{8\sqrt{5}}{4\sqrt{5}} = \frac{6}{\sqrt{5}}-2 = \frac{6\sqrt{5}}{5}-2.$$

Thus, $x-y = (2\sqrt{5}-2)-\left(\frac{6\sqrt{5}}{5}-2\right)$

$$= 2\sqrt{5}\cancel{-2}-\frac{6\sqrt{5}}{5}\cancel{+2} = \frac{10\sqrt{5}}{5}-\frac{6\sqrt{5}}{5} = \boxed{\frac{4\sqrt{5}}{5}.}$$