Barycentric Coordinates

Name:______________________________ Date:_______________

How would you approach this problem? Think of as many ways as you can.

1. In $\triangle ABC$, D is on $\overline{AB}$ and E is on $\overline{BC}$. Let $F = \overline{AE} \cap \overline{CD}$, $AD = 3$, $DB = 2$, $BE = 3$, and $EC = 4$. Find $\frac{EF}{FA}$ in simplest form.

The barycentric coordinate system is a powerful tool that can be used to solve triangle problems. Today we will look at how these coordinates can be applied to find ratios of segment lengths, calculate areas, and determine whether points are collinear. More topics for investigation into barycentric coordinates include perpendicularity of segments, the distance formula, and the circle equation.

For a fixed triangle $\triangle ABC$, define the point P corresponding to the barycentric coordinates $(x : y : z)$ as the center of mass of the system created when masses x, y, and z are placed on A, B, and C, respectively.

2. What point corresponds to $(2 : 2 : 0)$?
3. What point corresponds to $(5 : 5 : 5)$? What are other barycentric coordinates for this point?

Because of scaling, many sets of coordinates can correspond to a single point. By choosing a specific set of coordinates, we gain more information to work with, and the coordinates have other geometric significance. We choose coordinates so that $x + y + z = 1$, and call these coordinates the normalized barycentric coordinates. Then each point has a unique representation with normalized barycentric coordinates, so we can write $P = (x, y, z)$. Find the normalized barycentric coordinates of these points.

4. The midpoints M_a, M_b, M_c.
5. The centroid G.

The normalized barycentric coordinates (x, y, z) of P have these useful geometric interpretations in addition to the center of mass definition.

(i) Let $[*]$ denote the area of $*$. Then $x = \frac{[\triangle PBC]}{[\triangle ABC]}$, $y = \frac{[\triangle APC]}{[\triangle ABC]}$, and $z = \frac{[\triangle ABP]}{[\triangle ABC]}$.

(ii) Extend $\overline{AP}$, $\overline{BP}$, and $\overline{CP}$ to the opposite edges to points D, E, and F, respectively. Then $x = \frac{PD}{AD}$, $y = \frac{PE}{BE}$, and $z = \frac{PF}{CF}$. Furthermore, $\frac{BD}{DC} = \frac{z}{y}$, $\frac{CE}{EA} = \frac{x}{z}$, and $\frac{AF}{FB} = \frac{y}{x}$.

(iii) If each point is a position vector, $\vec{P} = x\vec{A} + y\vec{B} + z\vec{C}$.

Find the normalized barycentric coordinates of these points. Try to find and justify the coordinates of each point with more than one of these properties.

6. The vertices of the original triangle, A, B, and C.
7. The incenter I. (In terms of the side lengths a, b, and c.)
8. The circumcenter O. (In terms of the angles A, B, and C.)
9. Point D on $\overline{AB}$ such that $AD = 3$ and $BD = 4$.
10. Point H_a on $\overline{BC}$ such that $\overline{AH_a} \perp \overline{BC}$. (In terms of side lengths and angles.)

Here are some problems that can be solved with normalized barycentric coordinates.

11. (1988 AIME 12) Let P be an interior point of $\triangle ABC$ and extend lines from the vertices through P to the opposite sides. Let $AP = a$, $BP = b$, $CP = c$, and the extensions from P to the opposite sides all have length d. If $a + b + c = 43$ and $d = 3$, then find abc.

12. (1989 AIME 15) Point P is inside $\triangle ABC$. Line segments $\overline{APD}$, $\overline{BPE}$, and $\overline{CPF}$ are drawn with D on $\overline{BC}$, E on $\overline{CA}$, and F on $\overline{AB}$. Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$, and $CF = 20$, find the area of triangle ABC.

As with a Cartesian coordinate system, we can use equations to represent sets of points. In particular, we can look at lines. The equation of a line is $ux + vy + wz = 0$, where u, v, and w are real numbers, unique up to scaling. (Take a look at the next section for hints of a proof.)

13. Write an equation of the line parallel to $\overline{BC}$ that passes through G.

14. Write an equation of the line containing the median through A.

15. (2007/8 BMO-1 5) Let P be an internal point of triangle ABC. The line through P parallel to $\overline{AB}$ meets $\overline{BC}$ at L, the line through P parallel to $\overline{BC}$ meets $\overline{CA}$ at M, and the line through P parallel to $\overline{CA}$ meets $\overline{AB}$ at N. Prove that $\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} \leq \frac{1}{8}$, and locate the position of P in triangle ABC when equality holds.

We can also use barycentric coordinates to compute area. For points $P = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$, and $R = (x_3, y_3, z_3)$, we have

$$\frac{[PQR]}{[ABC]} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

(To prove this, first look at the simpler case where $R = A$, and show the area is $y_1 z_2 - y_2 z_1$ by using the normalized barycentric coordinates of P in fixed triangle $\triangle ABQ$. Then generalize for B and C, and weave together several of these triangles for $\triangle PQR$. You may need to use $x_i + y_i + z_i = 1$ multiple times.) How can this area relationship be used to show that P, Q, and R are collinear?

16. Let D and E be the feet of the altitudes from A and B respectively, and P and Q be the intersections of the angle bisectors $\overline{AI}$ and $\overline{BI}$ with $\overline{BC}$ and $\overline{CA}$, respectively. Show that D, I, E are collinear if and only if P, O, Q are.

17. (2012 ARML Team 7) Given noncollinear points A, B, C, segment $\overline{AB}$ is trisected by points D and E, and F is the midpoint of segment $\overline{AC}$. $\overline{DF}$ and $\overline{BF}$ intersect $\overline{CE}$ at G and H, respectively. If $[DEG] = 18$, compute $[FGH]$.

Answers

1. Mass points solution. Assign masses of 8 to A, 12 to B, and 9 to C. Then E has a mass of $12+9=21$, which means $\frac{EF}{FA}=\frac{8}{21}$.

   Menelaus' Theorem solution. In $\triangle ABE$, points D, C, and F are collinear and are on sides $\overline{AB}$, $\overline{BE}$, and $\overline{EA}$ respectively. Then $-1=\frac{AD}{DB}\cdot\frac{BC}{CE}\cdot\frac{EF}{FA}=\frac{3}{2}\cdot\frac{7}{-4}\cdot\frac{EF}{FA}$. Then $\frac{EF}{FA}=\frac{8}{21}$.

   Parallel line solution. Find G on $\overline{AB}$ so that $\overline{EG}\parallel\overline{CD}$. In $\triangle BCD$, $\overline{EG}\parallel\overline{CD}$ so $\frac{BE}{EC}=\frac{BG}{GD}$ and $GD=\frac{8}{7}$. Also, in $\triangle AGE$, $\overline{DF}\parallel\overline{GE}$ so $\frac{AD}{DG}=\frac{AF}{FE}$ and $\frac{EF}{FA}=\frac{8/7}{3}=\frac{8}{21}$.

2. Putting equal masses on A and B creates a system where the center of mass is halfway between A and B, so M_c corresponds to $(2:2:0)$.

3. When equal masses are placed on the vertices, the center of mass of the system is the centroid of the triangle, G. Any barycentric coordinates with all three coordinates equal will correspond to the centroid.

4. $M_a=\left(0,\frac{1}{2},\frac{1}{2}\right), M_b=\left(\frac{1}{2},0,\frac{1}{2}\right), M_c=\left(\frac{1}{2},\frac{1}{2},0\right)$.

5. $G=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$.

6. $A=(1,0,0), B=(0,1,0), C=(0,0,1)$.

7. $I=\left(\frac{a}{a+b+c},\frac{b}{a+b+c},\frac{c}{a+b+c}\right)$.

8. $O=\left(\frac{\sin A\cos A}{\sin A\cos A+\sin B\cos B+\sin C\cos C},\frac{\sin B\cos B}{\sin A\cos A+\sin B\cos B+\sin C\cos C},\frac{\sin C\cos C}{\sin A\cos A+\sin B\cos B+\sin C\cos C}\right)$.

9. $D=\left(\frac{4}{7},\frac{3}{7},0\right)$.

10. $H_a=\left(0,\frac{b\cos C}{a},\frac{c\cos B}{a}\right)$.

11. The normalized barycentric coordinates of P are $\left(\frac{d}{d+a},\frac{d}{d+b},\frac{d}{d+c}\right)$ by (ii). Then $\frac{d}{d+a}+\frac{d}{d+b}+\frac{d}{d+c}=1$. Multiply through by $(d+a)(d+b)(d+c)$ and group terms according to powers of d to obtain $2d^3+(a+b+c)d^2-abc=0$, from which $abc=441$.

12. The normalized barycentric coordinates of P are $\left(\frac{6}{12},\frac{3}{12},z\right)$ by (ii). Since the coordinates sum to 1, $z=\frac{1}{4}$, $PC=15$, and $BD=DC$ by (ii). Reflect P over D to P' to form $\triangle CDP'$, which is congruent to $\triangle BDP$ by SAS. Then $\triangle PP'C$ is a 9-12-15 triangle and has equal area to $\triangle PBC$. Since $\frac{[PBC]}{[ABC]}=\frac{1}{2}$ by (i), $[ABC]=2\cdot[PBC]=2\cdot 54=108$.

13. All points P on this line are the same distance from $\overline{BC}$, so the heights (and areas) of all such $\triangle PBC$ are equal. From (i), P must satisfy the equation $x=\frac{1}{3}=\frac{1}{3}(x+y+z)$, or $2x-y-z=0$.

14. Points $A=(1,0,0)$ and $M_a=\left(0,\frac{1}{2},\frac{1}{2}\right)$ are on this line, so $y=z$, or $y-z=0$.

15. Let $P=(x,y,z)$. Since $\overline{PL}\parallel\overline{AB}$, P and L have the same z-coordinate. L is on $\overline{BC}$, so its x-coordinate is 0. Finally, L's y-coordinate must be $x+y$ because the three coordinates sum to 1.

Then $L = (0, x+y, z)$ and by (ii), $\frac{LC}{BL} = \frac{x+y}{z}$. Similarly, $\frac{MA}{CM} = \frac{y+z}{x}$ and $\frac{NB}{AN} = \frac{z+x}{y}$. By the AM-GM inequality, we have $\frac{x+y}{z} \geq \frac{2\sqrt{xy}}{z}$. Then

$$\begin{aligned}\frac{LC}{BL} \cdot \frac{MA}{CM} \cdot \frac{NB}{AN} &= \frac{x+y}{z} \cdot \frac{y+z}{x} \cdot \frac{z+x}{y} \\ &\geq \frac{2\sqrt{xy}}{z} \cdot \frac{2\sqrt{yz}}{x} \cdot \frac{2\sqrt{zx}}{y} \\ &= 8.\end{aligned}$$

Inverting the inequality, we obtain $\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} \leq \frac{1}{8}$ with equality when $x = y = z$, that is, when P is the centroid.

16. *Note: As stated, it is false for triangles with right angle C. The correct statement considers only acute triangles. See 1997 IMO Shortlist and the 32nd German Federal Mathematical Competition Second Round, Q 4 (2001-2002).*

    Use $D = \left(0, \frac{b\cos C}{a}, \frac{c\cos B}{a}\right)$, $E = \left(\frac{a\cos C}{b}, 0, \frac{c\cos A}{b}\right)$, $P = \left(0, \frac{b}{b+c}, \frac{c}{b+c}\right)$, $Q = \left(\frac{a}{a+c}, 0, \frac{c}{a+c}\right)$, $I = \left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right)$, and $O = \left(\frac{2R^2}{bc}\cos A, \frac{2R^2}{ca}\cos B, \frac{2R^2}{ab}\cos C\right)$. We have

$$\begin{aligned}\frac{[DIE]}{[ABC]} &= \begin{vmatrix} 0 & \frac{b\cos C}{a} & \frac{c\cos B}{a} \\ \frac{a}{a+b+c} & \frac{b}{a+b+c} & \frac{c}{a+b+c} \\ \frac{a\cos C}{b} & 0 & \frac{c\cos A}{b} \end{vmatrix} \\ &= \frac{c\cos C}{a+b+c}\left(\cos C - \cos A - \cos B\right)\end{aligned}$$

and

$$\begin{aligned}\frac{[POQ]}{[ABC]} &= \begin{vmatrix} 0 & \frac{b}{b+c} & \frac{c}{b+c} \\ \frac{2R^2}{bc}\cos A & \frac{2R^2}{ca}\cos B & \frac{2R^2}{ab}\cos C \\ \frac{a}{a+c} & 0 & \frac{c}{a+c} \end{vmatrix} \\ &= \frac{2R^2}{(a+c)(b+c)}\left(\cos C - \cos A - \cos B\right).\end{aligned}$$

If D, I, and E are collinear, then $[DIE] = 0$, $\cos C - \cos A - \cos B = 0$, and $[POQ] = 0$, that is P, O, and Q are collinear. The converse is also true by similar reasoning.

17. The order of the points on $\overline{AB}$ must be $\overline{AEDB}$, as $\overline{ADEB}$ creates parallel $\overline{CE}$ and $\overline{DF}$ (and no G). We have $D = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$, $E = \left(\frac{2}{3}, \frac{1}{3}, 0\right)$, and $F = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$. Then $\overline{BF}$ is given by $x - z = 0$, $\overline{CE}$ by $x - 2y = 0$, and $\overline{DF}$ by $2x - y - 2z = 0$. So $H = \left(\frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right)$ and $G = \left(\frac{4}{9}, \frac{2}{9}, \frac{1}{3}\right)$. Then $\frac{[DEG]}{[ABC]} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{4}{9} & \frac{2}{9} & \frac{1}{3} \end{vmatrix} = -\frac{1}{9}$ and $\frac{[FGH]}{[ABC]} = \begin{vmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{4}{9} & \frac{2}{9} & \frac{1}{3} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{vmatrix} = \frac{1}{90}$. (The negative area represents a reversed orientation.) So $[FGH] = \frac{1}{10}[DEG] = \frac{9}{5}$.