Name:________________________ Date:____________

How would you approach this problem? Think of as many ways as you can.

1. In $\triangle ABC$, D is on $\overline{AB}$ and E is on $\overline{BC}$. Let $F = \overline{AE} \cap \overline{CD}$, $AD = 3$, $DB = 2$, $BE = 3$, and $EC = 4$. Find $\frac{EF}{FA}$ in simplest form.

The barycentric coordinate system is a powerful tool that can be used to solve triangle problems. Today we will look at how these coordinates can be applied to find ratios of segment lengths, calculate areas, and determine whether points are collinear. More topics for investigation into barycentric coordinates include perpendicularity of segments, the distance formula, and the circle equation.

For a fixed triangle $\triangle ABC$, define the point P corresponding to the barycentric coordinates $(x : y : z)$ as the center of mass of the system created when masses x, y, and z are placed on A, B, and C, respectively.

2. What point corresponds to $(2 : 2 : 0)$?

3. What point corresponds to $(5 : 5 : 5)$? What are other barycentric coordinates for this point?

Because of scaling, many sets of coordinates can correspond to a single point. By choosing a specific set of coordinates, we gain more information to work with, and the coordinates have other geometric significance. We choose coordinates so that $x + y + z = 1$, and call these coordinates the normalized barycentric coordinates. Then each point has a unique representation with normalized barycentric coordinates, so we can write $P = (x, y, z)$. Find the normalized barycentric coordinates of these points.

4. The midpoints M_a, M_b, M_c.

5. The centroid G.

The normalized barycentric coordinates (x, y, z) of P have these useful geometric interpretations in addition to the center of mass definition.

(i) Let $[*]$ denote the area of $*$. Then $x = \frac{[\triangle PBC]}{[\triangle ABC]}$, $y = \frac{[\triangle APC]}{[\triangle ABC]}$, and $z = \frac{[\triangle ABP]}{[\triangle ABC]}$.

(ii) Extend $\overline{AP}$, $\overline{BP}$, and $\overline{CP}$ to the opposite edges to points D, E, and F, respectively. Then $x = \frac{PD}{AD}$, $y = \frac{PE}{BE}$, and $z = \frac{PF}{CF}$. Furthermore, $\frac{BD}{DC} = \frac{z}{y}$, $\frac{CE}{EA} = \frac{x}{z}$, and $\frac{AF}{FB} = \frac{y}{x}$.

(iii) If each point is a position vector, $\vec{P} = x\vec{A} + y\vec{B} + z\vec{C}$.

Find the normalized barycentric coordinates of these points. Try to find and justify the coordinates of each point with more than one of these properties.

6. The vertices of the original triangle, A, B, and C.

7. The incenter I. (In terms of the side lengths a, b, and c.)

8. The circumcenter O. (In terms of the angles A, B, and C.)

9. Point D on $\overline{AB}$ such that $AD = 3$ and $BD = 4$.

10. Point H_a on $\overline{BC}$ such that $\overline{AH_a} \perp \overline{BC}$. (In terms of side lengths and angles.)

Here are some problems that can be solved with normalized barycentric coordinates.

11. (1988 AIME 12) Let P be an interior point of $\triangle ABC$ and extend lines from the vertices through P to the opposite sides. Let $AP = a$, $BP = b$, $CP = c$, and the extensions from P to the opposite sides all have length d. If $a + b + c = 43$ and $d = 3$, then find abc.

12. (1989 AIME 15) Point P is inside $\triangle ABC$. Line segments $\overline{APD}$, $\overline{BPE}$, and $\overline{CPF}$ are drawn with D on $\overline{BC}$, E on $\overline{CA}$, and F on $\overline{AB}$. Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$, and $CF = 20$, find the area of triangle ABC.

As with a Cartesian coordinate system, we can use equations to represent sets of points. In particular, we can look at lines. The equation of a line is $ux + vy + wz = 0$, where u, v, and w are real numbers, unique up to scaling. (Take a look at the next section for hints of a proof.)

13. Write an equation of the line parallel to $\overline{BC}$ that passes through G.

14. Write an equation of the line containing the median through A.

15. (2007/8 BMO-1 5) Let P be an internal point of triangle ABC. The line through P parallel to $\overline{AB}$ meets $\overline{BC}$ at L, the line through P parallel to $\overline{BC}$ meets $\overline{CA}$ at M, and the line through P parallel to $\overline{CA}$ meets $\overline{AB}$ at N. Prove that $\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} \le \frac{1}{8}$, and locate the position of P in triangle ABC when equality holds.

We can also use barycentric coordinates to compute area. For points $P = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$, and $R = (x_3, y_3, z_3)$, we have

$$\frac{[PQR]}{[ABC]} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

(To prove this, first look at the simpler case where $R = A$, and show the area is $y_1 z_2 - y_2 z_1$ by using the normalized barycentric coordinates of P in fixed triangle $\triangle ABQ$. Then generalize for B and C, and weave together several of these triangles for $\triangle PQR$. You may need to use $x_i + y_i + z_i = 1$ multiple times.) How can this area relationship be used to show that P, Q, and R are collinear?

16. Let D and E be the feet of the altitudes from A and B respectively, and P and Q be the intersections of the angle bisectors $\overline{AI}$ and $\overline{BI}$ with $\overline{BC}$ and $\overline{CA}$, respectively. Show that D, I, E are collinear if and only if P, O, Q are.

17. (2012 ARML Team 7) Given noncollinear points A, B, C, segment $\overline{AB}$ is trisected by points D and E, and F is the midpoint of segment $\overline{AC}$. $\overline{DF}$ and $\overline{BF}$ intersect $\overline{CE}$ at G and H, respectively. If $[DEG] = 18$, compute $[FGH]$.