NYC Math Team Practice - February 5, 2016

Challenging Problems in Algebra

1.) Let a, b, c, d be positive integers such that $\frac{a}{c} = \frac{b}{d} = \frac{3}{4}$ and $\sqrt{a^2+c^2} - \sqrt{b^2+d^2} = 15$. Compute $ac + bd - ad - bc$

2.) For all real x, find the minimum value of $\sqrt{x^2+9} + \sqrt{x^2-22x+137}$

3.) For all real x, compute the value of x that minimizes $\sqrt{x^4-x^2+1} + \sqrt{x^4-3x^2-6x+13}$ and compute the minimum value.

4.) Let z and $z+1$ be complex numbers such that z and $z+1$ are both n^{th} complex roots of unity. If n is a multiple of 5, compute the minimum value of $n + z^3$.

5.) Given the points $P_1, P_2, \dots, P_{10}$ in the complex plane. You are also given that

$$|P_1 - P_2| = |P_2 - P_3| = \ldots = |P_9 - P_{10}| = |P_{10} - P_1|$$

Also, P_1 is located at $(1, 0)$ and P_6 at $(3, 0)$. If P_n is the point with coordinates (x_n, y_n), then compute $(x_1 + y_1 i)(x_2 + y_2 i) \ldots (x_{10} + y_{10} i)$.

6.) Let α, β, γ be three real numbers such that:

$$2\cos(\alpha) + 3\cos(\beta) + 4\cos(\gamma) = 0$$
$$2\sin(\alpha) + 3\sin(\beta) + 4\sin(\gamma) = 0$$

Compute the maximum possible value of $4\cos\alpha + 9\cos\beta + 16\cos\gamma$

7.) For positive integer n, define S_n to be the minimum value of the sum

$$\sum_{k=1}^{n} \sqrt{(2k-1)^2 + (a_k)^2}$$

where $a_1, a_2, \dots, a_k$ are positive real numbers whose sum is 17.
There is a unique positive integer n for which S_n is also an integer. Compute n.

8.) For real numbers α and β, compute the minimum value of

$$(2\cos\alpha + 5\sin\beta - 8)^2 + (2\sin\alpha + 5\cos\beta - 15)^2$$

9.) Let w and z be complex numbers such that $|w| = 1$ and $|z| = 10$. Let $\theta = \arg\left(\frac{w-z}{z}\right)$. The maximum possible value of $\tan^2\theta$ can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$. (Note that $\arg(w)$, for $w \neq 0$, denotes the measure of the angle that the ray from 0 to w makes with the positive real axis in the complex plane.

10.) Given positive real numbers x, y, z that satisfy the following system of equations:

$$x^2 + xy + y^2 = 1$$
$$y^2 + yz + z^2 = 4$$
$$z^2 + zx + x^2 = 5$$

Compute $x + y + z$

11.) Let a, b, c be complex numbers such that $a^2 + b^2 + c^2 = ab + bc + ca$, $|a - b| = 2\sqrt{3}$, and $|a + b + c| = 21$. Compute $|a|^2 + |b|^2 + |c|^2$

12.) A person walks a very strange path starting from the point P_0 as follows:

The person walks 1 inch east to point P_1 and for $j \geq 1$, once the person reaches point P_j, they will turn 30° counterclockwise and then walks $j + 1$ inches straight to point P_{j+1}.

When the person reaches P_{2015}, they are $a\sqrt{b} + c\sqrt{d}$ inches away from their starting position, where $a, b, c, d \in \mathbb{N}$ and b and d are not divisible by the square of any prime.
Compute $a + b + c + d$

Answers:

1.) 108

2.) $\sqrt{170}$

3.) The minimum value is $\sqrt{10}$ and it occurs when $x = \frac{1+\sqrt{37}}{6}$

4.) 31

5.) 1023

6.) $\sqrt{31}$

7.) 12

8.) 100

9.) 100

10.) $\sqrt{5 + 2\sqrt{3}}$

11.) 159

12.) 2024