Solutions to Individual Round Questions from 1/16/15

By Nicholas Beasley

I1) If $(319, b, 481)$ is a Pythagorean triple, with $b<481$, find b.

Since $b<481$, we know that $319^2+b^2=481^2$ by the Pythagorean theorem. Subtracting 319^2 from both sides yields:

$$b^2=481^2-319^2.$$

This can be factored, since it is a difference of squares

$$b^2=(481+319)(481-319)$$

$$b^2=800(162)$$

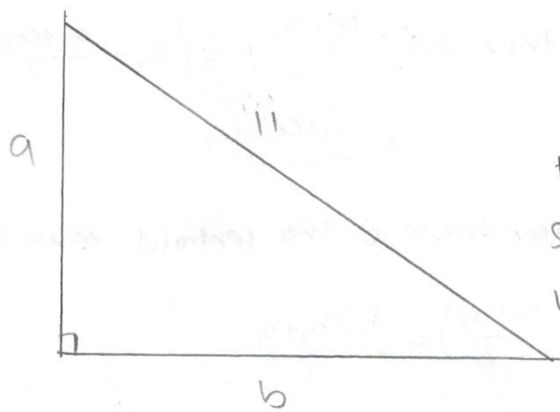
We know that 81 is a perfect square. If we divide 162 by 2 and multiply 800 by 2, we get:

$$b^2=1600(81)$$

$$b=40(9)$$

$$\boxed{b=360}$$

I2) The hypotenuse of a right triangle is 11 and its area is 21. Find its perimeter.



Let one leg have length a and the other leg have length b. We know that $a^2+b^2=11^2$ by the Pythagorean Theorem and that $\frac{ab}{2}=21$, since the area of a triangle is $\frac{1}{2}$(base)(height). We need to find $a+b$ in order to find the perimeter of the triangle. However,

$$(a+b)^2=a^2+b^2+2ab.$$

We know $a^2+b^2=121$ and $ab=42$, so

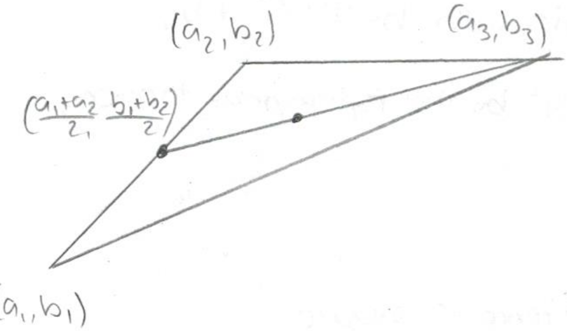
$$(a+b)^2=121+2(42)$$

$$(a+b)^2=205$$

$$(a+b)=\sqrt{205}.$$

The perimeter is $a+b+11=\boxed{11+\sqrt{205}}$

I3) On the coordinate plane: $A(12,20)$, $B(57,100)$ $C(-30,-63)$. Find the coordinates of the intersection of the medians of $\triangle ABC$.



* Centroid = intersection of the medians

We will prove that in general, for a triangle with coordinates (a_1,b_1), (a_2,b_2), and (a_3,b_3), the intersection of the medians lies at

$$\left(\frac{a_1+a_2+a_3}{3}, \frac{b_1+b_2+b_3}{3}\right).$$

Without loss of generality, pick the side of the triangle containing (a_1,b_1) and (a_2,b_2). The midpoint of this side is at $\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right)$. We now consider the segment connecting $\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right)$ and (a_3,b_3). The intersection of the medians split the median into a 1:2 ratio. The distance from $\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right)$ to the centroid* is therefore $\frac{1}{3}$ of the distance from $\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right)$ to (a_3,b_3). The x-coordinate of the centroid must then be $\frac{a_1+a_2}{2}+\frac{1}{3}\left(a_3-\frac{a_1+a_2}{2}\right)$

$$= \frac{a_1+a_2+a_3}{3},$$

while the y-coordinate of the centroid must be

$$\frac{b_1+b_2}{2}+\frac{1}{3}\left(b_3-\frac{b_1+b_2}{2}\right) = \frac{b_1+b_2+b_3}{3}.$$

Thus, the centroid lies at $\left(\frac{a_1+a_2+a_3}{3}, \frac{b_1+b_2+b_3}{3}\right)$, which in this problem is the point $\left(\frac{12+57+(-30)}{3}, \frac{20+100+(-63)}{3}\right)$

$$= \boxed{(13,19)}$$

I4) A class with 11 girls and n boys visited a library. Each child checked out the same number of books. Together they checked out n^2+9n-2 books from the library. How many boys are there in the class?

The number of books checked out per child is $\dfrac{\text{the number of books}}{\text{the number of children}}$

$$= \frac{n^2+9n-2}{n+11}$$

This must be an integer because books cannot be fractional.

$$\begin{array}{r|l} & n-2 \\ \hline n+11 & n^2+9n-2 \\ & -(n^2+11n) \\ \hline & -2n-2 \\ & -(-2n-22) \\ \hline & 20 \end{array}$$

$$\frac{n^2+9n-2}{n+11} = n-2+\frac{20}{n+11}$$

We know that n and 2 are integers, so $n-2$ is an integer. Therefore, in order for $\frac{n^2+9n-2}{n+11}$ to be an integer, $\frac{20}{n+11}$ must be an integer. This only occurs if $n+11$ is a divisor of 20. Since $n \geq 0$ (there cannot be negative boys), $n+11 \geq 11$. The only divisor of 20 greater than 11 is 20, so $n+11=20$

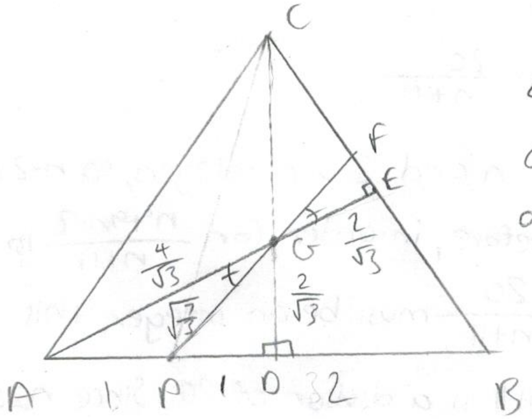
$n=9$,

and there are $\boxed{n=9}$ boys in the class.

15) In $\triangle ABC$, P is on $\overline{AB}$, and $AP:PB = 1:3$, Find the ratio into which the line through P and the centroid of $\triangle ABC$ divides $\overline{BC}$

* Note: This problem can probably be done with mass points, but this is my solution from practice:

Since the question does not specify anything about $\triangle ABC$, we can assume that it is an equilateral triangle, for convenience (this is a useful principle for math contests). We can also assume that $AP=1$. Therefore, $PB=3$.



We draw the altitude CD to AB. Since $\triangle ABC$ is equilateral, CD is also a median and contains the centroid, G. Since CD is a median, $DB=2$ and $PD = 3-2 = 1$.

In $\triangle AGD$, $\angle A = 30°$, so $GD = \frac{2}{\sqrt{3}}$ and $AG = \frac{4}{\sqrt{3}}$. In $\triangle PGD$, by the Pythagorean Theorem $GP = \sqrt{\frac{7}{3}}$. We now use the Law of Cosines in $\triangle AGP$.

$$AP^2 = AG^2 + GP^2 - 2AG(GP)\cos AGP$$

$$1^2 = \left(\frac{4}{\sqrt{3}}\right)^2 + \left(\sqrt{\frac{7}{3}}\right)^2 - 2\left(\frac{4}{\sqrt{3}}\right)\left(\sqrt{\frac{7}{3}}\right)\cos AGP$$

$$\cos AGP = \frac{5}{2\sqrt{7}}$$

Now, draw the line through P and the centroid intersect BC at F. $\angle AGP = \angle EGF$, since they are vertical angles. Therefore, $\cos\angle EGF = \frac{5}{2\sqrt{7}}$. Also, $GE = \frac{1}{2}AG = \frac{2}{\sqrt{3}}$, since the centroid splits medians into a 2:1 ratio. We can then solve for GF, which is $\frac{4\sqrt{7}}{5\sqrt{3}}$. Finally, the Pythagorean Theorem in

I5 (cont.) $\triangle EFG$ gives $GF^2 = EF^2 + EG^2$

$$\left(\frac{4\sqrt{7}}{5\sqrt{3}}\right)^2 = EF^2 + \left(\frac{2}{\sqrt{3}}\right)^2$$

$$EF^2 = \frac{112}{75} - \frac{4}{3}$$

$$EF^2 = \frac{4}{25}$$

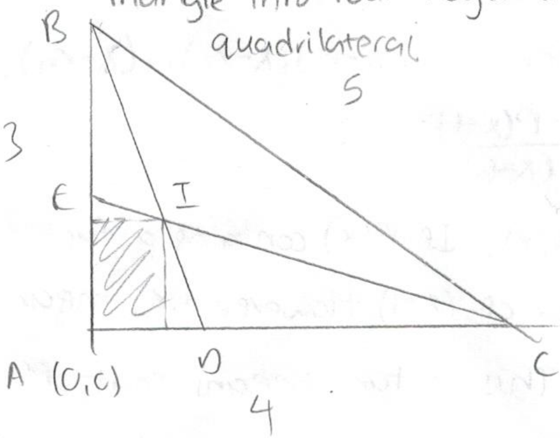
$$EF = \frac{2}{5}.$$

We now find the ratio $CF:BF$, since that is what we are looking for.

$$\frac{CF}{BF} = \frac{CE-EF}{BE+EF}$$

$$= \frac{2-\frac{2}{5}}{2+\frac{2}{5}} = \boxed{\frac{2}{3}}$$

I6) The angle bisectors of the acute angles of a 3-4-5 right triangle split the triangle into four regions. Compute the area of the region that is a quadrilateral.

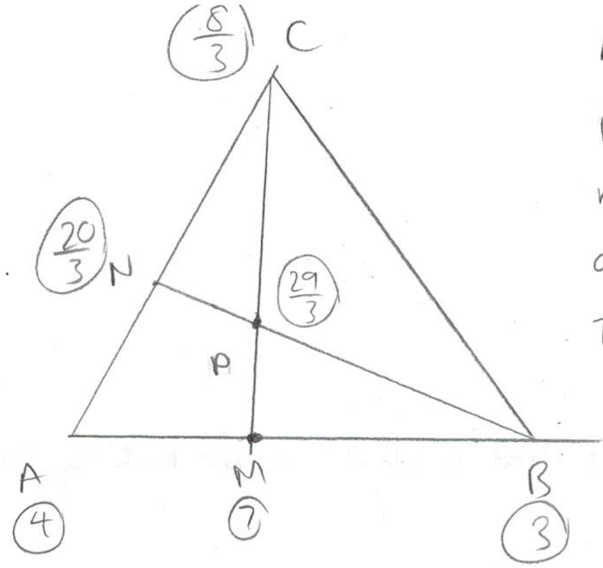


Place the triangle on the coordinate plane with the right angle at $(0,0)$. Then B is at $(0,3)$ and C is at $(4,0)$. Let BD and CE be angle bisectors. By the Angle Bisector Theorem, $AE = \frac{4}{3}$ and $AD = \frac{3}{2}$. Therefore, E is at $(0, \frac{4}{3})$ and AD is at $(\frac{3}{2}, 0)$.

The equation of line CE is $y = -\frac{1}{3}x + \frac{4}{3}$, while the equation of line BD is $y = -2x + 3$. The intersection of these lines, point I, occurs at $(1,1)$. The area of $ADIE$ can then be found by dividing the region into a square (shaded) and two triangles. The total area is $1 + \frac{1}{2}\left(\frac{1}{3}\cdot 1\right) + \frac{1}{2}\left(\frac{1}{2}\cdot 1\right)$

$$= \boxed{\frac{17}{12}}$$

I7)



Assign a mass of 4 to A. The mass at B must then be 3 and the mass at C must be $\frac{8}{3}$. This makes the mass at M 7 and the mass at N $\frac{20}{3}$. The mass at P is $\frac{29}{3}$. The ratio CP:PM is then $\frac{7}{\frac{8}{3}}$

$= \boxed{\frac{21}{8}}$

Problem: In $\triangle ABC$, M is on $\overline{AB}$, $AM:MB = 3:4$, N is on $\overline{AC}$, $AN:NC = 2:3$, and $\overline{BN}$ intersects $\overline{CM}$ at P. Compute CP:PM

I8) $P(x)$ is a monic polynomial (its leading coefficient is 1) which satisfies the identity $(x-6)P(x) = x \cdot P(x-1)$. Find $P(-1)$

Since $P(x)$ is monic, it can be written as $(x-r_1)(x-r_2)\ldots(x-r_n)$.

$$(x-6)P(x) = x \cdot P(x-1) \rightarrow P(x) = \frac{x \cdot P(x-1)}{(x-6)}.$$

This means that x is a factor of $P(x)$. If $P(x)$ contains a factor of x, then $P(x-1)$ contains a factor of $(x-1)$. However, this means that $P(x)$ has a factor of $(x-1)$. This in turn means that $P(x)$ has a factor of $(x-2)$. Similar reasoning continues until we have that $P(x)$ has a factor of $x-5$. Then $P(x-1)$ has a factor of $(x-6)$. However, this cancels out with the $x-6$ in the denominator, so $P(x)$ does not have a factor of $x-6$. In addition, if $P(x)$ had any other factors, they would not be contained in $\frac{x \cdot P(x-1)}{(x-6)}$.

There fore, $P(x) = x(x-1)(x-2)(x-3)(x-4)(x-5)$

$P(-1) = -1(-2)(-3)(-4)(-5)(-6)$

$P(-1) = \boxed{720}$