Solutions to Individual Round Problems from 2/6/15 Nicolas Naing

I1) Sally and Tommy did 20 assignments each, and received a grade from 2 to 5. For each assignment, Tommy got as many 5's as Sally got 4's, as many 4's as Sally got 3's, as many 3's, as Sally got 2's, and as many 2's as Sally got 5's. The average grade was the same. How many 5's did Sally get?

Solution: Let

a = number of 2's Tommy received
b = number of 3's Tommy received
c = number of 4's Tommy received
d = number of 5's Tommy received

We can organize Tommy's and Sally's scores like so:

Tommy

Score	# of Times
2	a
3	b
4	c
5	d

Sally

Score	# of Times
2	b
3	c
4	d
5	a

Their total scores are the same since their average grade was the same, so

$$2a+3b+4c+5d = 2b+3c+4d+5a$$
$$b+c+d = 3a$$

Since each person did 20 assignments, $a+b+c+d=20$ and

$$20-a=3a$$
$$a=5$$

I2) If $a, b,$ and c are the roots of $3x^3 - 3kx^2 + 3x + 2 = 0$ where k is a real number, and if

$$a^3 + b^3 + c^3 = 0,$$

find all real k.

Solution: Since $a, b,$ and c are the roots of $3x^3 - 3kx^2 + 3x + 2 = 0$,

$$3a^3 - 3ka^2 + 3a + 2 = 0$$
$$3b^3 - 3kb^2 + 3b + 2 = 0$$
$$3c^3 - 3kc^2 + 3c + 2 = 0$$

Adding these three equations gives

$$3(a^3+b^3+c^3) - 3k(a^2+b^2+c^2) + 3(a+b+c) + 6 = 0$$

(1) $$(a^3+b^3+c^3) - k(a^2+b^2+c^2) + (a+b+c) + 2 = 0$$

By Vieta's formulas on the equation $3x^3 - 3kx^2 + 3x + 2 = 0$,

$$a+b+c = \frac{-(-3k)}{3} = k$$

$$a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ac)$$

$$ab+ac+bc = \frac{3}{3} = 1$$

$$a^2+b^2+c^2 = k^2 - 2$$

Substituting these and $a^3+b^3+c^3 = 0$ into (1) gives

$$-k^3 + 2k + k + 2 = 0$$
$$-k^3 + 3k + 2 = 0$$
$$k^3 - 3k - 2 = 0$$

By the Rational Root Theorem, $k = -1$ is a root, so

$$(k+1)(k^2 - k - 2) = 0$$
$$(k+1)^2(k-2) = 0$$
$$k = -1, 2$$

13) Let $x = \frac{(3-\sqrt{5})}{2}$. Compute the exact value of $x^4 + \frac{1}{x^4}$.

Solution: Consider the equation $x + \frac{1}{x} = y$ for some value y. We can square this equation to get

$$x^2 + 2 + \frac{1}{x^2} = y^2$$

$$x^2 + \frac{1}{x^2} = y^2 - 2$$

We can square this again to get

$$x^4 + 2 + \frac{1}{x^4} = (y^2-2)^2$$

$$x^4 + \frac{1}{x^4} = (y^2-2)^2 - 2$$

Substituting $\frac{(3-\sqrt{5})}{2}$ for x gives

$$y = \frac{(3-\sqrt{5})}{2} + \frac{2}{3-\sqrt{5}}$$

$$= \frac{(3-\sqrt{5})}{2} + \frac{2(3+\sqrt{5})}{9-5}$$

$$= \frac{(3-\sqrt{5})}{2} + \frac{6+2\sqrt{5}}{4}$$

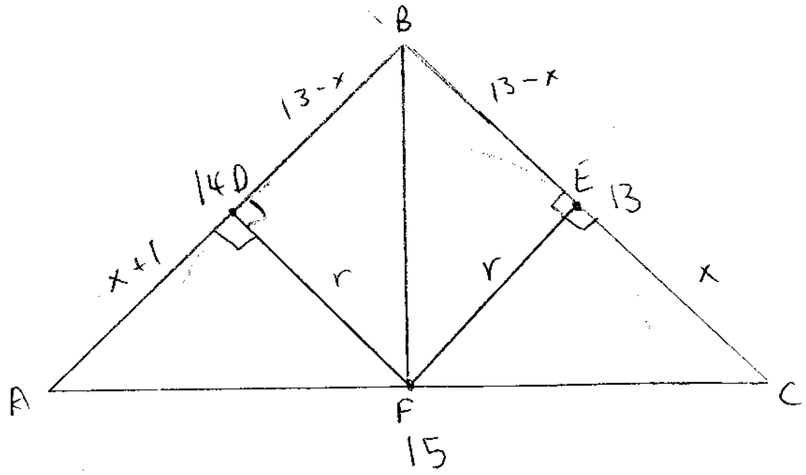
$$= \frac{6-2\sqrt{5}+6+2\sqrt{5}}{4}$$

$$= \frac{12}{4}$$

$$y = 3$$

Using this, $x^4 + \frac{1}{x^4} = (3^2-2)^2 - 2 = (7)^2 - 2 = 47$

I4) Compute the greatest possible radius of a semicircle contained entirely within a triangle with sides 13, 14, and 15.



The semicircle with the greatest area is tangent to the triangle at two points. Let ABC be the given triangle where $BC = 13$, $AB = 14$, and $AC = 15$. Let the semicircle intersect $\overline{AB}$ at D and $\overline{BC}$ at E and let the center of the semicircle be F.

Let $CE = x$. Then $BE = 13 - x$. Since $\overline{BD}$ and $\overline{BE}$ are both tangent to the semicircle, $BD = BE = 13 - x$. Since $BD = 13 - x$ and $BD + DA = 14$, $DA = x + 1$.

We can split ΔABC into ΔADF, ΔDBF, ΔBFE, and ΔEFC, so

$$[\Delta ABC] = [\Delta ADF] + [\Delta DBF] + [\Delta BFE] + [\Delta EFC]$$

Since D and E are points of tangency, $FD \perp AB$ and $FE \perp BC$. Then

$$[\Delta ADF] = \frac{(x+1)r}{2} = \frac{xr + r}{2}$$
$$[\Delta DBF] = \frac{(13-x)r}{2} = \frac{13r - xr}{2}$$
$$[\Delta BFE] = \frac{(13-x)r}{2} = \frac{13r - xr}{2}$$
$$[\Delta EFC] = \frac{xr}{2}$$

Adding these areas gives $\dfrac{xr + r + 13r - xr + 13r - xr + xr}{2} = \dfrac{27r}{2}$, so

$$[\Delta ABC] = \frac{27r}{2}$$

By Heron's Formula,

$$[\Delta ABC] = \sqrt{21(8)(7)(6)} = \sqrt{3 \cdot 7 \cdot 2^3 \cdot 7 \cdot 3 \cdot 2} = \sqrt{2^4 \cdot 3^2 \cdot 7^2} = 84$$

Then $84 = \dfrac{27r}{2}$ and $r = \dfrac{168}{27} = \dfrac{56}{9}$