Problem 1. *Sally and Tommy did 20 assignments each, and received a grade from 2 to 5 for each assignment. Tommy got as many 5's as Sally got 4's, as many 4's as Sally got 3's, as many 3's as Sally got 2's, and as many 2's as Sally got 5's. The average grade was the same. How many 5's did Sally get?*

Solution. Let t_i denote the number of assignments Tommy did with grade i, and let s_i denote the number of assignments Sally did with grade i. We can simply naively note that since their averages are equal, we can let $t_2 = t_3 = t_4 = t_5 = s_2 = s_3 = s_4 = s_5 = 5$ to make them equal and make the total number of assignments add to 20, resulting in an answer of $\boxed{5}$.

Alternate Solution: The sum of the scores of the assignments must be equal, so we get that $5t_5 + 4t_4 + 3t_3 + 2t_2 = 4s_4 + 3s_3 + 2s_2 + 5s_5$, which is equivalent to $5s_4 + 4s_3 + 3s_2 + 2s_5 = 4s_4 + 3s_3 + 2s_2 + 5s_5$, or $s_4 + s_3 + s_2 - 3s_5 = 0$. Since $s_2 + s_3 + s_4 + s_5 = 20$, subtracting these two equations gives us $4s_5 = 20$, or $s_5 = 5$, which is the answer. □

Problem 2. *If a, b, and c are the roots of $3x^3 - 3kx^2 + 3x + 2 = 0$, where k is a real number, and if $a^3 + b^3 + c^3 = 0$, find all real k.*

Solution. By Viete's formulas,

$$\begin{aligned} a + b + c &= -\frac{a_2}{a_3} = -\frac{-3k}{3} = k \\ ab + bc + ac &= \frac{a_1}{a_3} = \frac{3}{3} = 1 \\ abc &= -\frac{a_0}{a_3} = -\frac{2}{3} \end{aligned}$$

a, b and c would satisfy $x^3 = kx^2 - x - \frac{2}{3}$. We can make a substitution:

$$\begin{aligned} a^3 + b^3 + c^3 &= \left(ka^2 - a - \frac{2}{3}\right) + \left(kb^2 - b - \frac{2}{3}\right) + \left(kc^2 - c - \frac{2}{3}\right) \\ &= k(a^2 + b^2 + c^2) - (a + b + c) - 2 \\ &= k((a + b + c)^2 - 2(ab + bc + ac)) - k - 6 \\ &= k(k^2 - 2) - k - 6 \\ &= k^3 - 3k - 2 = 0 \end{aligned}$$

Solving this equation (via the Rational Root Theorem) yields us roots of $k = -1, -1, 2$, so our answer is $k = \boxed{-1, 2}$. □

Problem 3. *Let $x = \frac{3-\sqrt{5}}{2}$. Compute the exact value of $x^4 + \frac{1}{x^4}$.*

Solution. Note that

$$\begin{aligned}
\frac{1}{x} &= \frac{2}{3-\sqrt{5}} \\
&= \frac{2}{3-\sqrt{5}}\left(\frac{3+\sqrt{5}}{3+\sqrt{5}}\right) \\
&= \frac{2(3+\sqrt{5})}{3^2-\sqrt{5}^2} \\
&= \frac{2(3+\sqrt{5}}{4} \\
&= \frac{3+\sqrt{5}}{2}
\end{aligned}$$

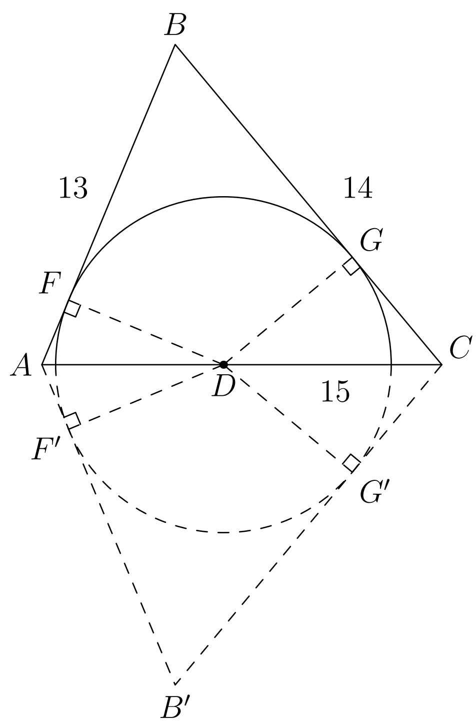
This means that $x + \frac{1}{x} = 3$.
Then see that (using a standard technique):

$$\left(x+\frac{1}{x}\right)^2 - 2 = x^2 + \frac{1}{x^2} \to x^2 + \frac{1}{x^2} = 7$$

Repeating this process yields $x^4 + \frac{1}{x^4} = 7^2 - 2 = \boxed{47}$. □

Problem 4. *Compute the greatest possible radius of a semicircle contained entirely within a triangle with sides* 13,14,*and* 15.

Solution. Below is a diagram:



From Heron's formula, we get that the area of ABC is $[ABC] = \sqrt{(21)(8)(7)(6)} = 84$. Denote the points of tangency to AB and BC as F and G, respectively. Reflect the triangle ABC across AC, as shown in the diagram. Let r be the radius of the circle, and let $GC = x$.

Then, $BG = 14 - x$, $BH = 14 - x$, and $AH = x - 1$. (We have analogous side lengths on the reflected triangle.) Drawing radii forms four quadrilaterals, and the sum of these areas equals the area of the entire quadrilateral:

$$\begin{aligned} [AFDF'] + [BFDG] + [CGDG'] + [B'F'G'D] &= [ABCB;] \\ (AF)(FD) + (BF)(FD) + (GC)(DG) + (B'F')(DF') &= 2[ABC] \\ (x-1)(r) + (14-x)(r) + (x)(r) + (14-x)(r) &= 2(84) \\ 27r &= 168 \\ r &= \frac{168}{27} = \boxed{\frac{56}{9}} \end{aligned}$$

□